Dr. Falk Hassler (falk.hassler@uwr.edu.pl)
Alex Swash (alex.swash@uwr.edu.pl)

Quantum Field Theory – Practice Exam

14^{th} of June 2024

Please fill in:

Name: ______________________

Matriculation Number: ______________________

Number of Sheets: ______________________

Instructions – Please read carefully:

- Please write your full name and matriculation number on each sheet you hand in.
- Use a separate sheet of paper for each problem.
- You have 120 minutes to answer the questions.
- No resources are allowed.
- Use a blue or a black permanent pen.

Do not write below this line.

Comments:

Question:	1	2	Total
Points:	25	25	50
Score:			

In the following, we take a look at (Quantum) Electrodynamics (Q)ED, in d dimensions with the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, \dots, -1)$. We start off with some basic properties of the classical theory and the path integral quantisation of the electromagnetic field in the first part of this exam. Afterward, we continue with selected aspects of its renormalisation.

1. Classical theory, quantisation, ... 25 points

Start with the Dirac Lagrangian

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi$$

for the spinor field ψ with $\partial\!\!\!/ = \gamma^\mu \partial_\mu$,

(a) (2 points) and obtain the Dirac equation by the least-action principle.

(b) (1 point) Show that the theory has a one continuous U(1) symmetry,

$$\psi \to \exp^{ie\chi} \psi \qquad \text{with} \qquad \chi \in [0, 2\pi)$$

denoting the parameter of the transformation, and a constant e.

(c) (2 points) Derive the Noether current

$$j^\mu = ie\bar{\psi}\gamma^\mu\psi$$

associated with this symmetry.

(d) (2 points) Verify that the current you found is conserved if the equations of motion are satisfied.

(e) (4 points) Gauge the U(1) symmetry of the theory by introducing the gauge potential A_μ, which transforms infinitesimally as

$$\delta A_\mu = \partial_\mu \chi \,.$$

Now $\chi = \chi(x)$ becomes coordinate dependent. Use the minimal coupling to couple the scalar field to an electromagnetic field. This implies that you have to add at least a term

$$-j^\mu A_\mu \tag{1}$$

to the original Lagrangian. Write down the full, gauged Lagrangian, including the kinetic term for the gauge field. Verify that it is invariant under the local U(1) symmetry we wanted to gauge.

(f) (3 points) Derive the classical equations of motion for all fields in the gauged theory (including the gauge field). You will find the relativistic form of the inhomogeneous Maxwell equations.

No we would like to quantise this theory in the path integral formalism. For the moment, we will only look at the free part. Interactions will be considered in the next problem. At the quadratic level, the spinor field ψ and the gauge potential A_μ decouple and can be treated independently.

(g) (3 points) To obtain the propagator for ψ, we need the differential operator

$$D_x\psi(x) = -\frac{\delta S}{\delta\bar{\psi}} \qquad \text{with} \qquad S = \int \mathrm{d}^d x \mathcal{L}$$

which gives the equations of motion, you already computed in (a), when setting it to zero. You know that the Feynman propagator $\Delta_F(x-y)$ is the Green's function

$$D_x \Delta_F(x-y) = -i\delta^d(x-y)\,.$$

Solve this equation by going to momentum space using

$$\tilde{\Delta}_F(p) = \int \mathrm{d}^d x \Delta_F(x) e^{-ipx}\,.$$

For the photon propagator, the situation is a bit more complicated. We have to isolate the divergence in the path integral for Z_0.

(h) (3 points) As a first step, we need to introduce a version of the δ-function in the path integral formalism. Prove by analogy with the finite-dimensional case, that the equation

$$1 = \int \mathcal{D}\alpha(x)\,\delta(G(\alpha))\det\left(\frac{\delta G(\alpha)}{\delta\alpha}\right)\,.$$

Using this expression, we can rewrite

$$Z_0 = \int \mathcal{D}A e^{iS[A]} = \int \mathcal{D}\alpha \int \mathcal{D}A \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) e^{iS[A]}\delta\left(G(A)\right) \tag{2}$$

with

$$A^\alpha_\mu = A_\mu + \tfrac{1}{e}\partial_\mu\alpha$$

and the Gauge fixing function

$$G(A) = \partial^\mu A_\mu - \omega(x)\,.$$

(i) (1 point) Explain why we can write in (2) $\delta\left(G(A)\right)$ instead of $\delta\left(G(A^\alpha)\right)$.

(j) (1 point) Show why we can pull out the determinant from the path integral.

The final trick is to average over all possible gauge choices aka $\omega(x)$. We do so with a Gaussian factor resulting in

$$Z_0 = N(\xi)\int \mathcal{D}\omega \exp\left(-i\int \mathrm{d}^d x\,\frac{\omega^2}{2\xi}\right) Z_0$$

with an appropriately fixed normalisation constant $N(\xi)$.

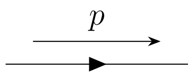
(k) (3 points) This integration can be performed easily because of the δ-function. As a result, you will obtain a quadratic effective action. Use this action to eventually compute the propagator for the gauge field A_μ following the same steps as in task (g).

2. ... and renormalisation. 25 points

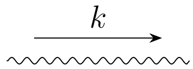
Consider the Lagrangian density for (spinor) QED in 4-dimensions ($d = 4$)

$$\mathcal{L} = i\bar{\psi}\not{D}\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

with $\not{D} = \gamma^\mu D_\mu$, $D_\mu = \partial_\mu + ieA_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In the following, we study this theory using the path integral formalism. Expanding the Lagrangian in terms of the gauge potential A_μ and the spinor field ψ to quadratic order, we found in the lecture the propagators



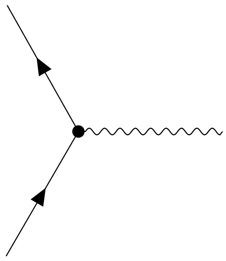
$$= i\frac{\not{p} + m}{p^2 - m^2}\,, \qquad \text{and}$$



$$= i\frac{g_{\mu\nu}}{k^2}\,,$$

for the electron and photon (in Feynman gauge), respectively.

(a) (1 point) Expand the Lagrangian to cubic order to obtain the photon-electron vertex. You should find



$$= -ie\gamma^\mu\,.$$

Please show explicitly how to obtain this result. It is only stated here because we will need it for the following task.

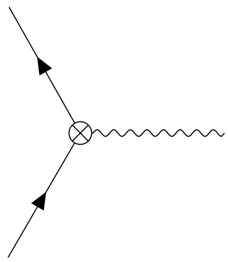
(b) (3 points) Next, compute the superficial degree of divergence by taking into account all Feynman rules above. *Hint: You should use Euler's formula for planar graphs* $1 = V - E + F$, *where* V *is the number of vertices,* E *denotes the number of edges, and* F *of faces.*

(c) (3 points) Use the result from task (b) to draw all the superficially divergent diagrams. Which of them are actually divergent, when you remember form the lecture that in QED only

- the electron mass and its field,
- the coupling to the electromagnetic field,
- and the polarisation of the vacuum

are renormalised?

(d) (3 points) Use these three divergent diagrams, to obtain the counter terms in the renormalised Lagrangian and their Feynman rules. In particular, you should get



$$= -ie\delta_{Z_e}\gamma^\mu\,.$$

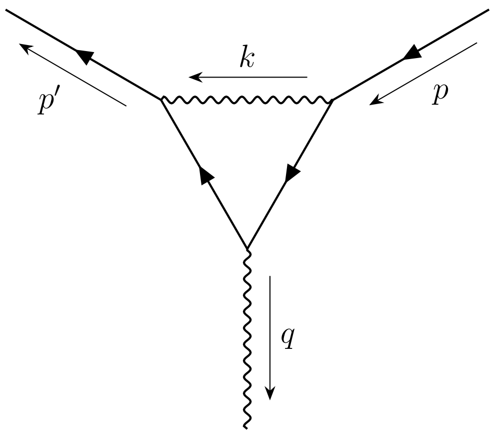
After this preparation, we can eventually compute the renomalized vertex at one-loop.

(e) (2 points) There is a useful identify for γ-matrices,

$$\gamma^\nu \gamma^\mu \gamma_\nu = (2-d)\gamma^\mu \,,$$

we need later on. Proof it, using the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ and $\eta^{\mu\nu}\eta_{\mu\nu} = d$.

(f) (4 points) Now, compute

$iV^\mu(p,p') =$ 

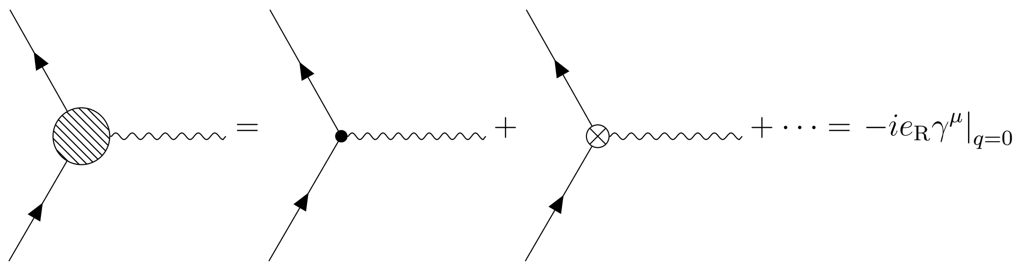
where $q = p - p'$ due to momentum conservation. Do not perform the loop integration at this stage, just apply the Feynman rules.

(g) (4 points) In general, you will that $iV^\mu(p,p')$ has the form

$$V^\mu(p,p') = f_1\gamma^\mu + f_2 p^\mu + f_3 p'^\mu + f_4 q^\mu$$

we only need f_1, because its divergence will be absorbed by the counter term we computed in (e). Furthermore, assume that the external states are on-shell, which implies $p^2 = p'^2 = m^2$. Compute f_1 and show that it can be written exclusively in terms of q^2. *Hint: f_1 is a Lorentz scalar, which only depends on p^μ and p'^μ. Therefore the only way these two can enter is the combination $p^\mu p'_\mu = p \cdot p'$. But at the same time, we know $q^2 = p^2 - 2p\cdot p' + p'^2$.*

To continue with the renormalisation, we need to fix a momentum at which we want to absorb the divergence into the counter term. A natural choice, because it corresponds to static electromagnetic interaction (Coulomb's law), is $q = 0$. Therefore, we are left with



with the renormalised charge e_R, defined by $(1+\delta_{Z_e})e_\mathrm{R} = e$.

(h) (4 points) Finally, set $q = 0$ and obtain the divergent part of $f_1(0)$, which we need to fix the counter term, by dimensional regularisation. *Hint: You should eventually find*

$$f_1(0) \sim \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\alpha k^2 + \beta m^2}{k^2 (k^2 - m^2)^2} \, .$$

The second term, proportional to β, is finite in $d = 4$. Hence you may drop it. Furthermore, the two following identities might come in handy:

- *Feynman parameter formula:*

$$\frac{1}{AB^2} = \int_0^1 \mathrm{d}x \frac{2(1-x)}{(xA + (1-x)B)^3} \, .$$

- *In $d = 4 - \epsilon$ dimensions*

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^{2a}}{(k^2 + \Delta)^b} = i \frac{1}{(4\pi)^{d/2}} \frac{1}{\Delta^{b-a-d/2}} \frac{\Gamma(a + d/2)\Gamma(b - a - d/2)}{\Gamma(b)\Gamma(d/2)}$$

where $\Gamma(\epsilon) = \epsilon^{-1} + \mathcal{O}(\epsilon)$.

(i) (1 point) Obtain the one-loop contribution to the counter term δ_{Z_e} by canceling the divergent part of $f_1(0)$ (in the minimal subtraction scheme).