Gauge/Gravity Duality, Winter 2024/25

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8. More Supersymmetry & String Theory (19 points)

To be discussed on Friday, 13th December, 2024 in the tutorial.
Please indicate your preferences until Sunday, 08/12/2024, 21:00:00 on the website.

Exercise 8.1: A toy model.

Consider the following 4d $\mathcal{N} = 1$ theory where the supersymmetric chiral multiplet has a complex scalar field ϕ and a left-handed Weyl fermion $\psi = \{\psi_\alpha\}_\alpha$ as components. These fields are massless and non-interacting. Therefore, the Lagrangian will be in the form

$$L = -\partial_\mu \phi^* \partial^\mu \phi - i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi.$$

a) (2 points) Show that L is invariant under the supersymmetry transformations

$$\delta_\epsilon \phi = \sqrt{2}\epsilon\psi,$$
$$\delta_\epsilon \psi_\alpha = \sqrt{2} i (\sigma^\mu \bar{\epsilon})_\alpha \partial_\mu \phi.$$

b) (3 points) Apply Noether's methods, which we discussed in the first lecture, to construct the associated supercurrent.

c) (4 bonus points) Construct the Weyl spinor Noether charges Q associated to the supersymmetry and show that

$$i\delta_\epsilon \phi = [\epsilon Q + \bar{\epsilon}\bar{Q}, \phi],$$
$$i\delta_\epsilon \psi = [\epsilon Q + \bar{\epsilon}\bar{Q}, \psi]\,.$$

Warning: This one might be a bit harder, but promises a lot of fun because you have to do canonical quantization for the charge Q. Saying that first you rewrite your result for Q in terms of the canonical fields and then assume canonical (anti-)commutation relations for them. After all this work, you will be rewarded with this beautiful algebra.

d) (4 points) Consider two transformations δ_ϵ, δ_η. Acting with $[\delta_\epsilon, \delta_\eta]$ on both ϕ and ψ, and knowing from the equations of motion that $\bar{\sigma}^\mu \partial_\mu \psi = 0$, show that the supersymmetry algebra reads

$$[\delta_\epsilon, \delta_\eta] = 2i(\eta\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\eta})\partial_\mu.$$

The commutator amounts, then, to a translation by $\eta\sigma^\mu\bar{\epsilon}$. Note that the fact that we had to use the equations of motion to prove closure of the algebra implies that the closure is only on-shell.

Exercise 8.2: Polyakov action.

a) (3 points) Show that the energy-momentum tensor

$$T_{\alpha\beta}(\sigma, z) = \frac{1}{2\pi\alpha'}\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{4\pi\alpha'}\eta_{\alpha\beta}\partial^\gamma X^\mu \partial_\gamma X_\mu,$$

is conserved and traceless.

b) (4 points) Introducing lightcone coordinates $\sigma^- = \tau - \sigma$, $\sigma^+ = \tau + \sigma$, one can write the mode expansion of X^μ as

$$X^\mu(\sigma^+, \sigma^-) = x^\mu + \alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} (\alpha_n^\mu e^{-in\sigma^-} + \tilde{\alpha}_n^\mu e^{-in\sigma^+}),$$

and of the stress-energy tensor as

$$T_{--} = \frac{1}{2\pi} \sum_n l_n e^{-in\sigma^-}.$$

Show that

$$l_n = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_m^\mu \alpha_{n-m}^\nu \eta_{\mu\nu},$$

where $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$.

c) (3 points) From the Poisson brackets

$$\left\{ \frac{\partial}{\partial \tau} X^\mu(\tau, \sigma), X^\nu(\tau, \sigma') \right\} = 2\pi \alpha' \eta^{\mu\nu} \delta(\sigma - \sigma'),$$

one finds that

$$\begin{aligned} \{\alpha_m^\mu, \alpha_n^\nu\} &= im\eta^{\mu\nu}\delta_{m+n}, \\ \{\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\} &= im\eta^{\mu\nu}\delta_{m+n}, \\ \{p^\mu, x^\nu\} &= \eta^{\mu\nu}. \end{aligned}$$

Verify that

$$\begin{aligned} \{l_m, l_n\} &= i(m-n) l_{m+n}, \\ \{l_m, X^\mu(\sigma^-, \sigma^+)\} &= ie^{im\sigma^-} \partial_{\sigma^-} X^\mu(\sigma^-, \sigma^+). \end{aligned}$$