Gauge/Gravity Duality, Winter 2024/25

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8. Supersymmetry (16 points)

To be discussed on Friday, 6th December, 2024 in the tutorial.
Please indicate your preferences until Monday, 02/12/2024, 21:00:00 on the website.

Exercise 8.1: Wigner classification.

Consider the Poincaré algebra

$$\begin{aligned}
[J_{\mu\nu}, J_{\rho\sigma}] &= i(\eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\nu\sigma}J_{\mu\rho}),\\
[J_{\mu\nu}, P_\rho] &= i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu),\\
[P_\mu, P_\nu] &= 0.
\end{aligned}$$

a) (3 points) A Casimir operator is a polynomial operator in the generators of the algebra that commutes with every generator. Show that the following two operators are Casimir operators for the Poincaré algebra:

$$\begin{aligned}
C^{(2)} &:= P^2 = P_\mu P^\mu \qquad \text{(quadratic Casimir = mass Casimir)},\\
C^{(4)} &:= W^2 = W_\mu W^\mu \qquad \text{(quartic Casimir = spin/helicity Casimir)},
\end{aligned}$$

where $W_\sigma = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}J^{\mu\nu}P^\rho$ is called the *Pauli-Lubanski pseudovector*. (Note: the full calculations are lenghty, therefore for the exercise just prove that W^2 commutes with P_μ and P^2 commutes with $J_{\mu\nu}$.)

b) (4 points) It is possible to show that the unitary irreducible representations (UIRREPS) of Poincaré are in correspondence with all possible particle states in a theory with Poincaré invariance. The classification of all these UIRREPS was made by Wigner in 1948. This means that we can classify all the particles in a Poincaré invariant theory according to their mass and spin/helicity eigenvalues. Since $p^2 = m$, it is easy to understand why the eigenvalues of the quadratic Casimir label masses of particles. For the quartic one, consider the following two cases:

- $p^2 > 0$. In this case we can choose a simple reference frame where $p_\mu = (p_0, 0, 0, 0)$ with $p_0 > 0$. Show that in this case the quartic Casimir labels the spin of a particle, $W^2 \propto m^2 S^2$, where we have defined the spin operator $S_i = \epsilon_{ijk}J_{jk}$.
- $p^2 = 0$ and $p_0 > 0$. In this case we can choose $p_\mu = (p_0, 0, 0, p_0)$. Show that in this case, when $W^2 = 0$, the components of the quartic Casimir label the helicity of a particle, $W_0^2 = W_3^2 \propto (S \cdot P)^2$ (remember that the helicity is the projection of S on the direction of P).

The other cases are not relevant for us because $p^2 = 0$ with $p_0 = 0$ gives just the vacuum, and $p^2 < 0$ is in general neglected since it gives rise to tachyonic particles.

c) (2 points) When one augments the Poincaré algebra to the super-Poincaré algebra, W^2 is not anymore a Casimir operator. One can, then, modify it in order to find a good Casimir operator. To understand how this modification works, show that one can rewrite

$$W^2 = C_{\mu\nu}C^{\mu\nu},$$

with $C_{\mu\nu} = W_\mu P_\nu - W_\nu P_\mu$. It is, then possible, to find a quartic Casimir modifying $C_{\mu\nu}$ as

$$\tilde{C}_{\mu\nu} = \tilde{W}_\mu P_\nu - \tilde{W}_\nu P_\mu,$$

with $\tilde{W}_\mu = W_\mu - \frac{1}{4}\bar{Q}_{a\dot{\alpha}}\bar{\sigma}_\mu^{\dot{\alpha}\beta}Q_\beta^a$.

d) (3 points) Show explicitly that W^2 is not a Casimir for the supersymmetry algebra computing a single bracket with it that does not vanish. Then show that the same bracket vanishes when one modifies W^2 as in the previous exercise.

Exercise 8.2: Fermion number operator.

Consider the following fermion number operator $\mathcal{N}_f$, acting on bosonic states $|B\rangle$ and fermionic states $|F\rangle$ as

$$\mathcal{N}_f|B\rangle = |B\rangle, \qquad \mathcal{N}_f|F\rangle = -|F\rangle.$$

a) (1 point) Knowing that the supersymmetry charges Q_α and $\bar{Q}_{\dot{\alpha}}$ transform a bosonic state into a fermionic state and vice versa, what is the anticommutator of them with $\mathcal{N}_f$?

b) (3 points) Using the cyclicity of the trace, show that $\mathrm{Tr}(\mathcal{N}_f\{Q_\alpha, \bar{Q}_{\dot{\beta}}\}) = 0$. Comparing this result with the expression of $\mathrm{Tr}(\mathcal{N}_f\{Q_\alpha, \bar{Q}_{\dot{\beta}}\})$ found employing the supersymmetry algebra, and knowing that the difference between bosonic degrees of freedom n_b and fermionic degrees of freedom n_f in a supersymmetry multiplet is given by $\mathrm{Tr}\,\mathcal{N}_f$, show that $n_b = n_f$.