Gauge/Gravity Duality, Winter 2024/25

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3. Dualities and General Relativity (20 points)

To be discussed on Friday, 18th October, 2024 in the tutorial.
Please indicate your preferences until Sunday, 13/10/2024, 21:00:00 on the website.

Exercise 3.1: Electromagnetic Duality

In this exercise we will discuss a simple and nice example of a duality arising in the classical regime between the electric and magnetic fields.

Consider the spaces of p-forms $\Omega^p(\mathcal{M})$, defined over a d-dimensional smooth manifold $\mathcal{M}$.

a) (2 points) Show that there is an isomorphism between $\Omega^p(\mathcal{M})$ and the space of $d-p$ forms $\Omega^{d-p}(\mathcal{M})$. *Hint: Show that both spaces have the same dimension.* This isomorphism is called the *Hodge $\star$-dual* and it is defined as

$$\star : \Omega^p(\mathcal{M}) \to \Omega^{d-p}(\mathcal{M}),$$

acting on a local basis as

$$\star(e^{a_1} \wedge \dots \wedge e^{a_p}) = \frac{1}{(d-p)!} \varepsilon_{b_1 \dots b_{d-p}}{}^{a_1 \dots a_p} e^{b_1} \wedge \dots \wedge e^{b_{d-p}}$$

with $e^a = e^a{}_i dx^i$ and $g_{ij} = \eta_{ab} e^a{}_i e^b{}_j$. Derive the action on a coordinate basis $\{dx^{\mu_i}\}|_{i=1,\dots,p}$, and on a p-form $\omega^{(p)}$.

b) (2 points) Show that the following equalities are true:

1) $\star 1 = dVol_{\mathcal{M}}$ with the volume form $dVol_{\mathcal{M}} = e^1 \wedge \dots \wedge e^d$,
2) $\star \star \omega^{(p)} = (-1)^{p(d-p)} \omega^{(p)}$ if the metric is Euclidean,
3) $\star \star \omega^{(p)} = (-1)^{p(d-p)+1} \omega^{(p)}$ if the metric is Lorentzian.

c) (3 points) **p-form electrodynamics.** Consider Maxwell theory in $d = 4$. The components of the vector potential A_μ are components of a 1-form $A \in \Omega^1$ valued in the Lie algebra $u(1)$, corresponding to the abelian gauge group of electrodynamics $U(1)$. Analogously, we can define its strength tensor as $dA = F \in \Omega^2$. From this follows

$$dF = 0, \tag{1}$$

a geometric constraint coming from the nilpotence of the exterior derivative: $dF = d^2 A = 0$. Show that the homogeneous Maxwell equations

$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{E} - \frac{1}{c} \partial_t \vec{B} = 0,$$

come from (1).
(*Hint: evaluate the components of the previous equation, remembering that the field thensor can be written as* $F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}$).

d) (3 points) Consider the operator $d^\dagger = \star d \star$, called the *formal adjoint* of d, along with the 4-current 1-form $j = j_\mu dx^\mu \in \Omega^1$. Show that the remaining two Maxwell equations

$$\frac{1}{c}\partial_t E + \nabla \times \vec{B} = \vec{j}, \qquad \nabla \cdot E = -j_0$$

are given by

$$d^\dagger F = j.$$

e) (2 points) **Electromagnetic duality.** Electromagnetic duality is the statement that the Maxwell equations (without sources) are invariant under the exchange $(E^i, B^i) \to (B^i, -E^i)$. This means that swapping the roles of electric and magnetic fields the physics still remain the same. To see what this means in terms of p-form electrodynamics, recall that in exercise 2.2 e) of the last problem set, we introduced the dual field strength $\tilde{F}^{\mu\nu}$ for a gauge theory, anticipating that this object was the Hodge dual of $F_{\mu\nu}$. Show that

$$\star F = \tilde{F}.$$

The duality, then, amounts to the substitution

$$F \to \tilde{F}.$$

This implies that electromagnetic duality is a duality mediated by the $\star$-Hodge operator. Apply this map to the Maxwell equations without sources ($dF = 0, d^\dagger F = 0$) to show that you still obtain Maxwell equations where the roles for the electric field and the magnetic field are swapped.

(*Hint: there is a very quick clever way to show it and a more lengthy one based on evaluating the equations component by component*).

Note that this duality can be lifted to the case in which one has sources when one considers not only an electric current j, but also a magnetic one k, modifying the equations $dF = 0$ to $dF = k$.

Exercise 3.2: Some properties of the Lie derivative and the Christoffel symbols

a) (2 points) Show that the Lie derivative of a vector field U along another vector field V may be rewritten as the Lie bracket of the two fields as

$$\mathcal{L}_V U^\mu = [V, U]^\mu = V^\nu \partial_\nu U^\mu - U^\nu \partial_\nu V^\mu,$$

and that

$$\mathcal{L}_V U^\mu = -\mathcal{L}_U V^\mu$$

holds.

b) (1 point) Show that in the case of a symmetric connection $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ we can replace ∂ with ∇ in the definition of the Lie derivative.

c) (2 points) Derive the explicit expression for the Christoffel symbols in terms of the metric tensor and its derivatives by taking into account the following properties

1. metricity,also called compatibility with the metric, $\nabla_\mu g_{\nu\lambda} = 0$, and
2. vanishing torsion, which just says that $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$.

d) (1 point) Show that

$$\mathcal{L}_V g_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu,$$

with the Levi-Civita connection computed in that last task. This is the transformation law for the metric under an infinitesimal coordinate transformation.

e) (2 points) Show that the relations

$$\Gamma^\mu_{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\nu \sqrt{-g},$$

and

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} V^\mu)$$

hold. We will need them to obtain the field equations for the Einstein-Hilbert action.