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13. Field-operator map. (19 points)

To be discussed on Friday, 31st January, 2025 in the tutorial.
Please indicate your preferences until Thursday, 23/01/2025, 21:00:00 on the website.

Exercise 13.1: Spherical harmonics. 3 points

Show that

$$\Box_{S^5} Y^l = -\frac{1}{L^2} l(l+4) Y^l,$$

where $\Box_{S^5}$ is the D'Alembert operator on S^5, L is the radius of S^5 and Y^l are spherical harmonics.

Exercise 13.2: Field-operator map.

a) (2 points) Starting from the action

$$\mathcal{S} = -\frac{C}{2} \int \mathrm{d}z \mathrm{d}^d x \sqrt{-g} \left(g^{mn} \partial_m \phi \partial_n \phi + m^2 \phi^2 \right),$$

show that for a scalar $\phi(z)$ in Poincaré coordinates, where the metric becomes

$$\mathrm{d}s^2 = g_{mn} \mathrm{d}x^m \mathrm{d}x^n = \frac{L^2}{z^2} (\mathrm{d}z^2 + \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu),$$

the action can be rewritten as

$$\mathcal{S} = -\frac{CL^{d-1}}{2} \int \mathrm{d}z \mathrm{d}^d x \frac{1}{z^{d+1}} \left(z^2 \partial_z \phi \partial_z \phi + m^2 L^2 \phi^2 \right). \tag{1}$$

b) (4 points) Consider the action (1). Show that near the boundary, where $\phi \sim z^\Delta$, it is finite when integrating from $z = 0$ to $z = \epsilon$ provided that $\Delta \geq d/2$.

c) (4 points) Breitenlohner–Freedman bound. In flat space, fields with negative m^2 have an upside-down potential which leads to an instability. In $(d+1)$-dimensional Anti-de Sitter space, however, scalar fields are still stable even for negative m^2 if their mass satisfies

$$m^2 L^2 \geq -\frac{d^2}{4},$$

provided the fluctuations have the asymptotic behaviour $\phi \sim z^\Delta$. This bound is called Breitenlohner–Freedman bound. In order to obtain it, start from (1), reparametrize it posing $y = \ln z$ and rescale the scalar as $\phi = z^{d/2} \varphi$. Show that, up to a boundary term, you obtain

$$\mathcal{S} = -\frac{CL^{d-1}}{2} \int \mathrm{d}y \mathrm{d}^d x \left[\partial_y \varphi \partial_y \varphi + \left(m^2 L^2 + \frac{1}{4} d^2 \right) \varphi^2 \right]. \tag{2}$$

This is the action of a scalar field φ in flat spacetime, with effective mass given by $m_e^2 L^2 = m^2 L^2 + \frac{1}{4} d^2$. Remembering that in flat spacetime a scalar field theory is consistent only if $m_e^2 \geq 0$, show that you obtain the Breitenlohner–Freedman bound.

d) (2 points) Starting from the action (1), integrating by parts and removing a finite boundary term, find the action

$$\mathcal{S} = -\frac{CL^{d-1}}{2}\int \mathrm{d}z\mathrm{d}^d x \frac{1}{z^{d+1}}\left(-z^2\phi\partial_z^2\phi + (d-1)z\phi\partial_z\phi + m^2L^2\phi^2\right). \tag{3}$$

Is this action the same as (1)? In other terms: for wich values of Δ does the boundary term vanish?

e) (4 points) Consider the action (3). Show that near the boundary, where $\phi \sim z^\Delta$, it is finite when integrating from $z = 0$ to $z = \epsilon$ provided that $\Delta \geq (d-2)/2$. This bound corresponds to the unitarity bound of a scalar field in a CFT, i.e., the bound that hasd to be satisfied in order to avoid negative norm states.