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10. More String Theory (12 points)

To be discussed on Tuesday, 7th January, 2025 in the tutorial.
Please indicate your preferences until Thursday, 02/01/2025, 21:00:00 on the website.

Exercise 10.1: Critical dimension and ζ-function regularisation

The mass-spectrum formula for the open string was given in the lecture by

$$M^2 = \frac{1}{\alpha'}(N-a)\,.$$

We now try to compute this constant by what is called ζ-function regularisation. The idea here is to regularise the sum

$$-\frac{D-2}{2}\sum_{p=1}^{\infty} p = a\,,$$

which arises from the normal ordering. To this end, we note that the Riemann ζ-function is given by

$$\zeta(s) = \sum_{p=1}^{\infty} p^{-s}\,.$$

a) (2 points) Prove that we can write the product of the ζ- and Γ-function,

$$\Gamma(s) = \int_0^\infty \mathrm{d}t\, e^{-t} t^{s-1}\,,$$

as

$$\Gamma(s)\zeta(s) = \int_0^\infty \mathrm{d}t \frac{t^{s-1}}{e^t - 1}$$

assuming that Re(s)>1.

b) (3 points) Verify the small t expansion

$$\frac{1}{e^t-1} = \frac{1}{t} - \frac{1}{2} + \frac{t}{12} + \mathcal{O}(t^2)\,,$$

(Hint: remember Laurent expansions.)

c) (2 points) and use it to show that for Re(s)>1

$$\Gamma(s)\zeta(s) = \int_0^1 \mathrm{d}t\, t^{s-1}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12}\right) + \frac{1}{s-1} + \frac{1}{2s} + \frac{1}{12(s+1)} + \int_1^\infty \mathrm{d}t \frac{t^{s-1}}{e^t-1}$$

holds.

d) (1 point) Explain why the right-hand side above is well defined also for Re(s)>-2. It follows that this right-hand side defines an analytic continuation of the left-hand side to Re(s)>-2.

e) (3 points) Recalling the pole structure of the Γ-function, use it to show that

$$\zeta(0) = -\frac{1}{2} \qquad \text{and} \qquad \zeta(-1) = -\frac{1}{12}\,.$$

Argue that the ζ-function regularisation implies that

$$\sum_{p=1}^{\infty} p = -\frac{1}{12}. \tag{1}$$

f) (1 point) Explain why the presence of a massless spacetime vectorial state in the spectrum of an open string implies $D = 26$.