The Generalized Cartan Geometry of α’-corrections

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Based on 2409.00176 and work in progress with

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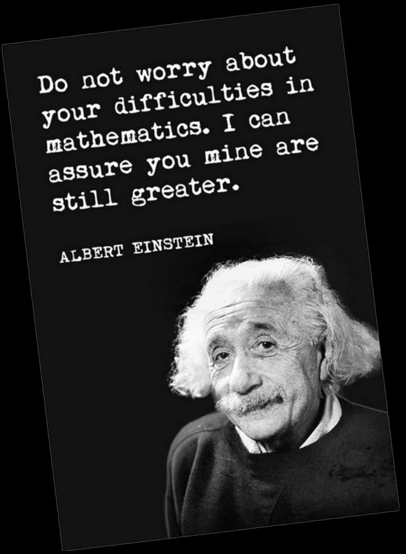
The Problem

- Einstein-Hilbert action is not renormalizable in d>2 → only EFT

$$S = \int \mathrm{d}x^d \sqrt{-g}\left(R + a_1 R^2 + a_2 R_{ij} R^{ij} + \ldots\right)$$



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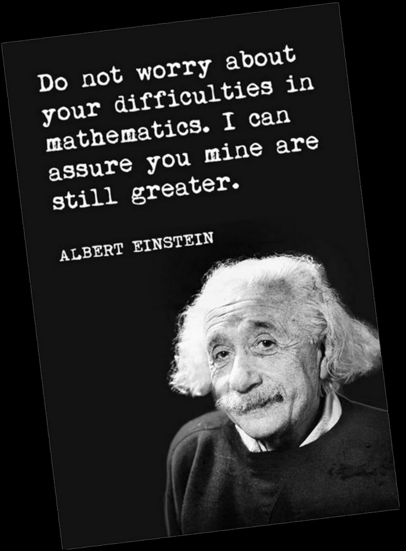
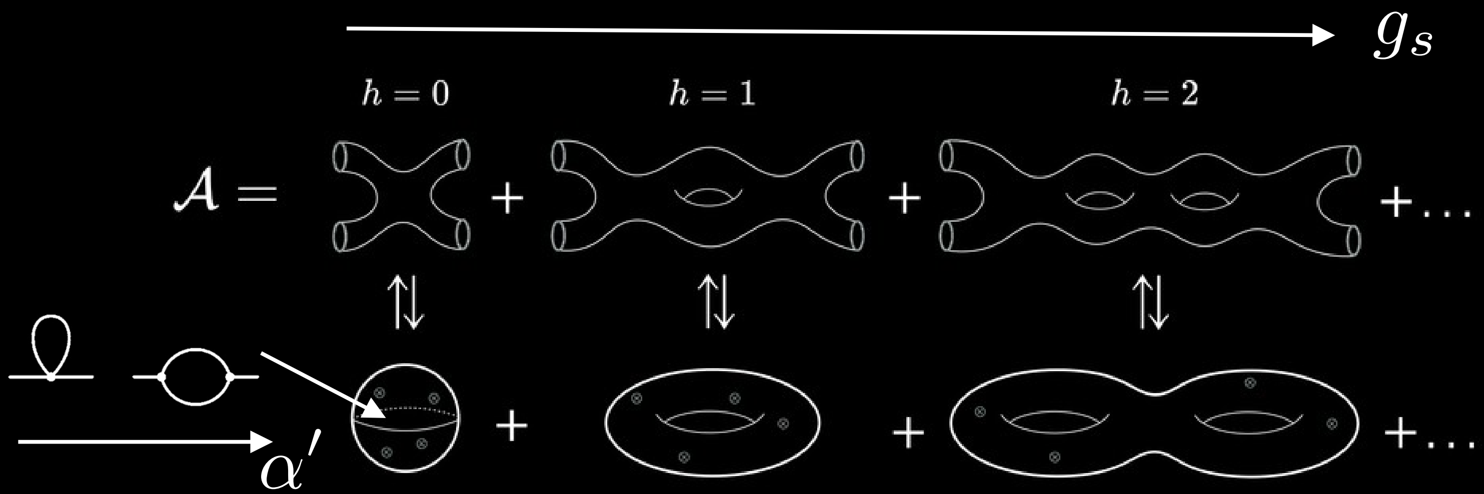
Question: How do we obtain all the coefficients ?

String Theory





NS/NS-sector @ leading order in α’

$$\widetilde{H}_{ijk} = H_{ijk} - \tfrac{3}{2} a \Omega^{(-)}_{ijk} + \tfrac{3}{2} b \Omega^{(+)}_{ijk}$$

$$S = \int \mathrm{d}x^d \sqrt{-g} e^{-2\phi} \Big(R + 4(\partial\phi)^2 - \tfrac{1}{12} \widetilde{H}^2 + \tfrac{a}{8} R^{(-)}_{ija}{}^{b} R^{(-)ij}{}_{b}{}^{a} + \tfrac{b}{8} R^{(+)}_{ija}{}^{b} R^{(+)ij}{}_{b}{}^{a} + \dots \Big)$$

$a = -\alpha, b = 0$	heterotic
$a = b = -\alpha'$	bosonic
$a = b = 0$	type II

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- 3 coefficients for terms with 2
- 8 coefficients for terms with 4
- 60 coefficients for terms with 6

derivatives

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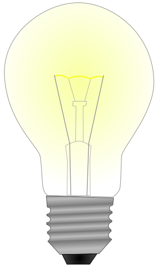
Too many terms. Nothing is known about > 8 derivatives.



A better approach:

Leverage symmetry to decrease number of possible terms.

Like diffeomorphisms, gauge-transformations and:

- SUSY
- Extended Generalized Lorentz Symmetry (today)
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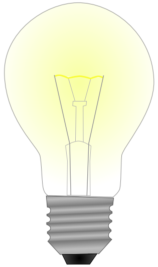

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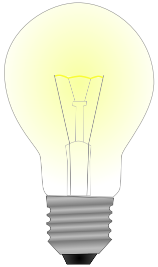
generalized frame

$$\mathrm{E}_A{}^I = \begin{pmatrix} e_a{}^i & e_a{}^j B_{ji} \\ 0 & e^a{}_i \end{pmatrix}$$



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$$E_A{}^I = \begin{pmatrix} e_a{}^i & e_a{}^j B_{ji} \\ 0 & e^a{}_i \end{pmatrix}$$

invariant under O(d)xO(d) ⊂ O(d,d)

$$\eta_{AB} = \begin{pmatrix} 0 & \delta_\alpha^\beta \\ \delta_\beta^\alpha & 0 \end{pmatrix} \quad H_{AB} = \begin{pmatrix} \delta_{ab} & 0 \\ 0 & \delta^{ab} \end{pmatrix}$$

Leading Symmetries and Action

$$\delta E^A{}_M = \mathbb{L}_\xi E^A{}_M + \Lambda^A{}_B E^B{}_M \, , \qquad \Lambda^A{}_B \in \mathrm{O}(d) \times \mathrm{O}(d)$$

generalized Lie derivative

1) diffeomorphisms (gravity)
2) gauge tranformation

generalized Lorentz transformation

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generalized flux $F_{ABC} = 3D_{[A}E_B{}^I E_{C]I}$ with $\mathrm{D}_A = E_A{}^i \partial_i$

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$$S = \int dx^d\, e^{-2d}\, \mathcal{R}$$

one unique invariant

$$\mathcal{R}(F_{ABC}, F_A, D_A, H_{AB})$$

We need a Factory for Invariants...

[Polacek, Siegel 13;
Butter 21;
Butter, FH, Pope, Zhang 23]

<u>Symmetries</u>

- gen. diff
- gen. Lorentz

Generalized Cartan Geometry*

<u>Invariants</u>

$\mathcal{R}, \ldots$

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gen. spin and affine connection, related by

$$\nabla_A E_B{}^M = 0$$

Covariant derivative:

$$\nabla_A E_B{}^M = E_A{}^N \partial_N E_B{}^M + \Omega_{AB}{}^C E_C{}^M - E_A{}^N \Gamma_{NL}{}^M E_B{}^L$$

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$$[\nabla_A, \nabla_B] V^C = R_{ABD}{}^C V^D + T_{AB}{}^D \nabla_D V^C$$

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2 connections are required: $\Omega_A^\alpha\,,\,\rho^{\alpha\beta}$ ← adjoint index of the gen. Lorentz group G_S

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$$\mathrm{O}(d+n,d+n) \to \mathrm{O}(d,d) \times G_{\mathrm{S}}\,, n = \dim(G_{\mathrm{S}})$$

$$\cup$$

$$G_{\mathrm{PS}} \supset G_{\mathrm{S}}$$

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<u>Cartan connection</u> $\theta(x) : T_x P \to \mathfrak{g}$

$$\theta^{\hat{a}}{}_{\hat{i}} = \begin{pmatrix} \delta^{\alpha}_{\mu} & \omega^{\alpha}{}_{i} \\ 0 & e^{a}{}_{i} \end{pmatrix}$$

$$\theta = t_{\hat{a}} \theta^{\hat{a}}{}_{\hat{i}} \mathrm{d}x^{\hat{i}}$$

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Cartan curvature

$$\Theta = -\mathrm{d}\theta + \frac{1}{2}[\theta, \theta]$$

$$T = -\mathrm{d}e + [\omega, e]$$

$$R = -\mathrm{d}\omega + \frac{1}{2}[\omega, \omega]$$

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Generalized Cartan curvature

$$\Theta_{\hat{A}\hat{B}} = -[\theta_{\hat{A}}, \theta_{\hat{B}}]_{\mathrm{D},\mathfrak{d}}$$

$\mathfrak{d}$-twisted Dorfman-bracket

Choosing G_{S} and G_{PS}

Objective:

1) fix all connections by
   1) gauge fixing
   2) torsion constraints

   in terms of the generalized frame (and its derivatives)
2) as few invariants as possible

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$$G_{\mathrm{S}} = \mathrm{O}(d+p) \times \mathrm{O}(d+q)$$
$$G_{\mathrm{PS}} = \mathrm{O}(d+p, d+q)$$

Recursive embedding of G_{S}

- G_{PS} is generated by $K_{AB}, {R_\alpha}^A, R_{\alpha\beta}$
- and G_{S} by $\mathrm{t}_\alpha = (t_{\overline{\alpha}} \quad t_{\underline{\alpha}})$

How to relate them ???

left and right factors of G_{S}

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- exponential growth of generators
- can be truncated at every order

Torsion contraints and gauge fixing

- Poláček-Siegel construction results one quantity (product):

The generalized Cartan curvature $\Theta_{\mathcal{ABC}}$ ← fundamental index of G_{PS}

- Remember, it contains all curvatures and torsions of the gen. connections

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- To fix them completely, we impose:

$$\Theta_{\overline{\mathcal{A}}\underline{\mathcal{BC}}} = \Theta_{\underline{\mathcal{A}}\overline{\mathcal{BC}}} = 0$$

Torsion contraint

$$\Omega_{\overline{a}}^{\overline{\alpha}} = \Omega_{\underline{a}}^{\underline{\alpha}} = \rho^{\overline{\alpha}\overline{\beta}} = \rho^{\underline{\alpha}\underline{\beta}} = 0$$

Gauge fixing of chiral/anti-chiral sector

Results

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There is a hidden symmetry in string theory which controls higher-derivative(α’)-corrections. How far can we push it?