massive representations

$p^\mu = (m, \underbrace{0,0,0}_{SO(3)=SU(2) \text{ little group}})$ $\sim S, S^3 \sim$ "z-component" (spin)

$\{Q^a_\alpha, \bar{Q}_{b\dot\beta}\} = 2m\,\delta^a_b(\sigma_0)_{\alpha\dot\beta} = 2m\,\delta^a_b \begin{pmatrix}1&0\\0&1\end{pmatrix}_{\alpha\dot\beta}$

$\Rightarrow \quad a^b_\alpha = \dfrac{Q^b_\alpha}{\sqrt{2m}}$ and $(a^+)^a_{\dot\alpha} = \dfrac{\bar{Q}^a_{\dot\alpha}}{\sqrt{2m}}$

lowering spin   raising spin $2^{2\mathcal{N}}$ states in multiplet

$\Rightarrow$ assuming vanishing central charge

Z^{ab} can be diagonalized to $Z^{ab} = \begin{pmatrix} \begin{matrix}0 & q_1\\ -q_1 & 0\end{matrix} & 0 & \\ 0 & \begin{matrix}0 & q_2\\ -q_2 & 0\end{matrix} & \\ & & \ddots \end{pmatrix}$

$\Rightarrow \quad \tilde{Q}^j_{\alpha\pm} = Q^{2j-1}_\alpha \pm (Q^{2j}_\alpha)^+ \quad j = 1, \dots, \mathcal{N}/2$

with $\{\tilde{Q}^i_{\alpha\pm}, (Q^j_{\beta\pm})^+\} = \delta^i_j \delta^\beta_\alpha \underbrace{(2m \pm q_j)}_{\geq 0 \text{ for unitary reps}}$

special case $|q_i| = 2m$ (charge, mass) $\hat{=}$ BPS bound

Bogomolnyi - Prasad - Sommerfield

K times $\rightsquigarrow$ $2\mathcal{N} - 2K$ creation operators and

$2^{2(\mathcal{N}-K)}$ states $=$ $1/2^K$ BPS multiplet

7.3. Supersymmetric field theories

(1.) free theory $\mathcal{L}_{free} = -\partial_\mu \phi^* \partial^\mu \phi - i\bar\Psi \bar\sigma^\mu \partial_\mu \Psi$

SUSY transformations $\begin{cases} \delta_\varepsilon \phi = \sqrt{2}\, \varepsilon\Psi, \\ \delta_\varepsilon \Psi_\alpha = \sqrt{2}\, i (\sigma^\mu \bar\varepsilon)_\alpha \partial_\mu \phi \end{cases}$ and $\{\varepsilon\Psi = \varepsilon^\alpha \Psi_\alpha$

$\rightsquigarrow$ details in the tutorial, in particular SUSY tr.

only close on-shell.

(2.) Wess-Zumino model ~ "Susy ϕ^3-theory"

$$\mathcal{L} = \mathcal{L}_{free} - \frac{1}{2} m \psi\psi - \frac{1}{2} m \bar{\psi}\bar{\psi} - g\phi\psi\psi - g^* \phi^* \bar{\psi}\bar{\psi}$$

$$\delta_\varepsilon \phi = \sqrt{2}\, \varepsilon \psi$$

non-linear in fields

$$\delta_\varepsilon \psi = \sqrt{2}\, i (\sigma^\mu \bar{\varepsilon}) \partial_\mu \phi - \sqrt{2} (m\phi^* + g\phi^{*2}) \varepsilon$$

Remark: • Finding Susy field theories is hard
⇝ use tools like Super Space formalism.
• We can combine super & conformal symmetry
⇝ SCFT (Super Conformal Field Theory)

7.4. d=4 max. Susy Yang-Mills

spin ≤ 1 ⇝ d=4 max. $\mathcal{N}=4$ $(-1 \to -1/2 \to 1/2 \to 1)$ helicity Q

⇝ $SU(4)$-R-symmetry and massless multiplet

	field	index	$SU(4)_R$ irrep
vector	A_μ ~ vector		1 (singlet)
Weyl fermion	λ^a_α ~ spinor	$a = 1, \dots, 4$	4 (fundamental)
Real scalars	ϕ^i	$i = 1, \dots, 6$	6 (antisymmetric $\begin{smallmatrix}\square\\\square\end{smallmatrix}$)

Properties: – gauge coupling $\tau = \frac{\vartheta}{2\pi} + i \frac{4\pi}{g^2_{YM}}$

– $SL(2,\mathbb{Z})$ symmetry $\tau' = \frac{a\tau + b}{c\tau + d}$

$a, b, c, d \in \mathbb{Z}$
$ad - bc = 1$

– SCFT (even after quantization)

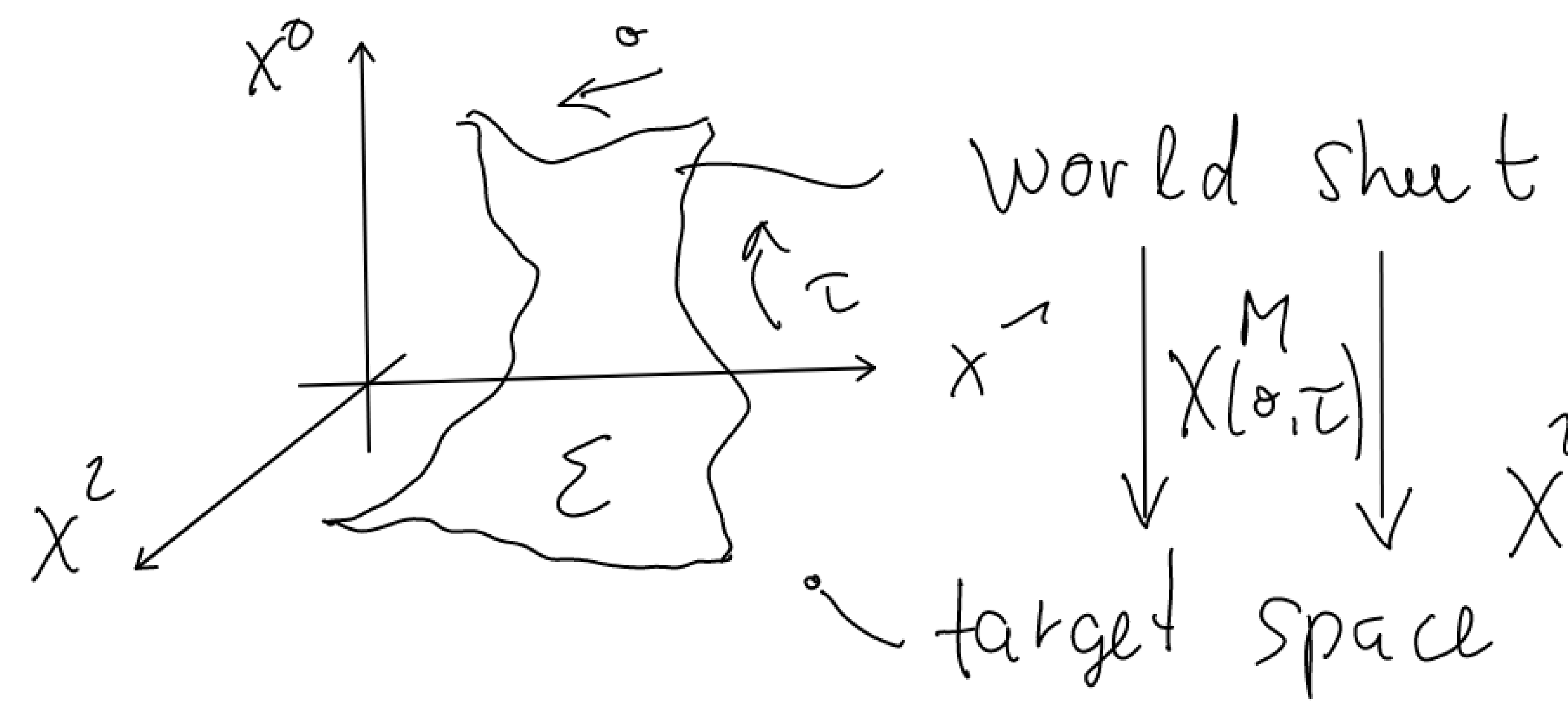
8. String theory

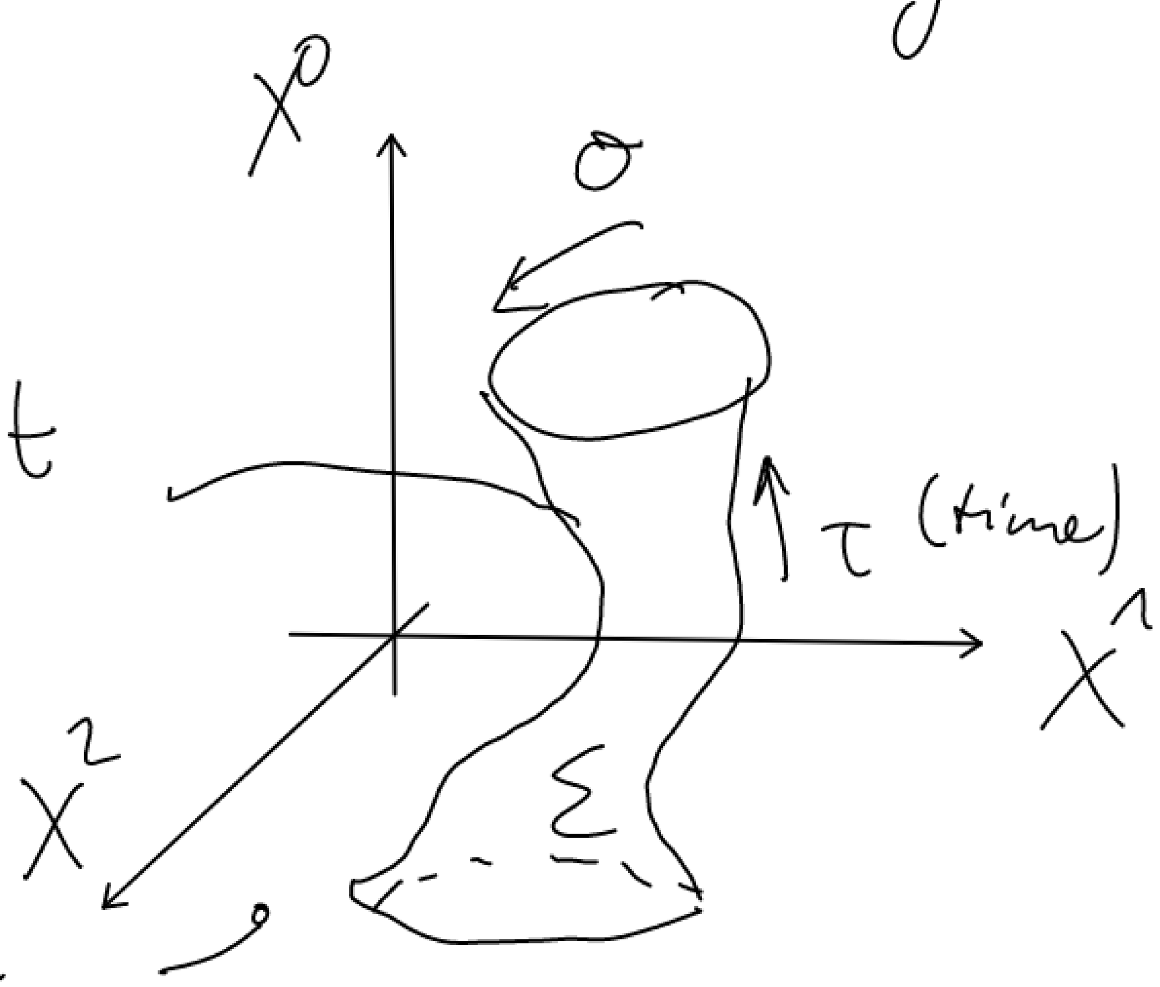
large N-limit $U(N)$ - (S)YM generating function
$\hat{=}$ string's $\rightsquigarrow$ closed strings describe gravity HOW?

8.1. From points to strings

- different vibration modes $\hat{=}$ diff. particles
- two types $\longrightarrow$ closed string $\to$ gravity

open string $\to$ gauge theory





action $S_{NG} = -\frac{1}{2\pi\alpha'}\int_\Sigma d^2\sigma \sqrt{-\det\left(\partial_\alpha X^M \partial_\beta X^N \eta_{MN}\right)}$

Nambu-Goto

(remember world line action for geodesic)

highly non-linear :-C $\rightsquigarrow$ make it linear by adding worldsheet metric $h_{\alpha\beta}$ $\rightsquigarrow$ Polyakov action

$$\boxed{S_P = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma \sqrt{-h}\, h^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N \eta_{MN}}$$

Integrating out $h_{\alpha\beta}$ by $\frac{\delta S_P}{\delta h_{\alpha\beta}} = 0 \rightsquigarrow S_{NG}$ (Virasoro constraints)

Symmetries:

global
- target space Poincaré
  $X^M \to X'^M = \Lambda^M{}_N X^N + a^M \qquad (\delta h_{\alpha\beta} = 0)$

local
- world sheet diff.'s (2d gravity)
- Weyl transformations

$$\left(h_{\alpha\beta}(\tau,\sigma) \to h'_{\alpha\beta}(\tau,\sigma) = e^{2\omega(\tau,\sigma)} h_{\alpha\beta}(\tau,\sigma)\right)$$

To quantize we gauge fix $h_{\alpha\beta} = e^{2\omega(\tau,\sigma)} \eta_{\alpha\beta}$
with $\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

= <u>conformal gauge</u>

$$S_P = \frac{1}{4\pi\alpha'} \int d^2\sigma \left(\partial_\tau X^M \partial_\tau X^N - \partial_\sigma X^M \partial_\sigma X^N\right) \eta_{MN}$$

with equations of motion $\left(\partial_\tau^2 - \partial_\sigma^2\right) X^M = \partial_+ \partial_- X^M = 0$
(light-cone coordinates $\sigma^\pm = \tau \pm \sigma$ with $\partial_\pm = \partial/\partial\sigma^\pm$)

Virasoro constraints from $\frac{\delta S_P}{\delta h^{\alpha\beta}} = 0$ become

$T_{++} = \partial_+ X^M \partial_+ X_M = 0$, $T_{--} = \partial_- X^M \partial_- X_M = 0$ and
$T_{+-} = T_{-+} = 0$

8.2. String spectrum in Minkowski space

solution of eom: $X^M(\tau,\sigma) = X_L^M(\sigma^+) + X_R^M(\sigma^-)$
left- and right moving

expansion into modes results in

$$X_L^M(\sigma^+) = \frac{\tilde{X}_0^M}{2} + \frac{\alpha'}{2}\tilde{p}^M \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\tilde{\alpha}_n^M}{n} e^{-in\sigma^+}$$

$$X_R^M(\sigma^-) = \frac{X_0^M}{2} + \frac{\alpha'}{2} p^M \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n^M}{n} e^{-in\sigma^-}$$

$X_0^M / \tilde{X}_0^M$ center of mass with momentum $p^M / \tilde{p}^M$

$\alpha_n^M / \tilde{\alpha}_n^M$ vibration modes X^M real

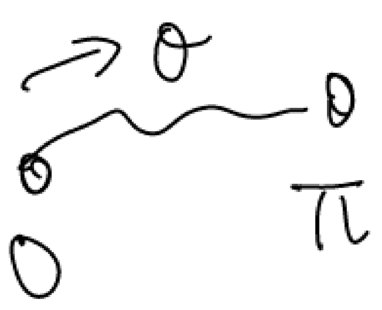
$\rightsquigarrow \alpha_{-n}^M = \left(\alpha_n^M\right)^*$ and same for $\tilde{\alpha}_n^M$

boundary conditions (b.c.)

- closed string: $X^\mu(\tau,0) = X^\mu(\tau,2\pi)$ & $\partial_o X^\mu(\tau,0) = \partial_o X^\mu(\tau,2\pi)$

  ⇝ $x_0^\mu = \tilde{x}_0^\mu$ and $p^\mu = \tilde{p}^\mu$

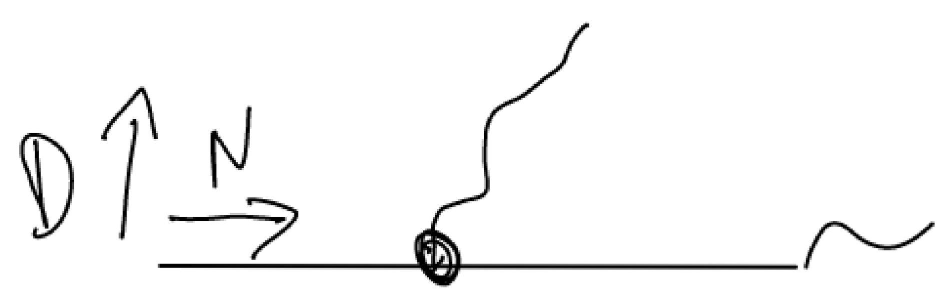
- open string with end points at $\bar{\sigma} = 0$ or π



Neumann b.c. $\partial_\sigma X^\mu(\tau,\bar{\sigma}) = 0$

Dirichlet b.c. $\delta X^\mu(\tau,\bar{\sigma}) = 0$

≙ end point is fixed $X^\mu(\tau,\bar{\sigma}) = \bar{x}_0^\mu$

combination of both 

~ D-bane

Canonical Quantization

1. Poisson brackets for $X_0^\mu, P^\mu; \alpha_n^\mu$ and $\tilde{\ }$ -"-

   ⇝ tutorial