Last time: Identify physical states with BRST cohomology

physics: States $|\phi\rangle$ with

$Q|\phi\rangle = 0$ and $N_g|\phi\rangle = 0$ are gauge invariant

These are the states we want!

hermitian

But if $|\phi\rangle = Q|\lambda\rangle$ $\quad \langle\lambda|Q^{+}Q|\lambda\rangle = \langle\lambda|Q^2|\lambda\rangle = 0$

null state

$\rightarrow$ remove it from the Hilbert space

$$\boxed{\mathcal{H} = \mathcal{H}_{phys} / \mathcal{H}_{null} = H^0 \text{ in BRST cohomology}}$$

Applied to ST $\quad Q = \sum_{m=-\infty}^{\infty} \left(c_{-m} L_m^X - \frac{1}{2} \sum_{n=-\infty}^{\infty} :c_{-m} c_{-n} b_{m+n}: \right)$

and $Q^2 = \frac{1}{2}\{Q, Q\} = \frac{1}{2} \sum_{m,n=-\infty}^{\infty} \left([L_m, L_n] - (m-n) L_{m+n} \right) c_{-m} c_{-n}$

$= 0$ for $D=26$ and $a=1$

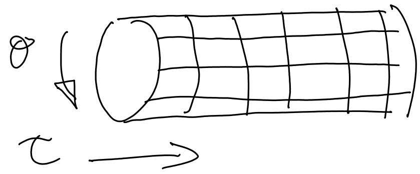
9. Conformal field theory

Idea: QFT + (global) conformal symmetry

$\rightarrow$ strongly constraints form of correlation functions

9.1. From cylinder to plane

Remember: world sheet of closed string

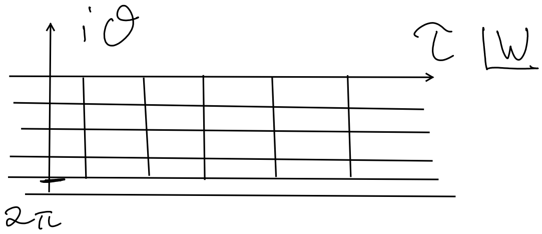


cylinder in Lorentzian signature

1.) Wick rotation $\tau \rightarrow -i\tau \rightarrow$ Euclidean signature

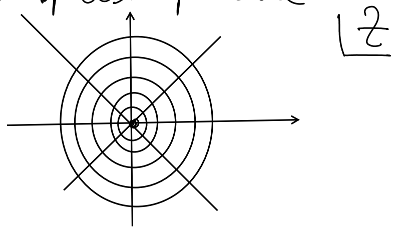
$\sigma^{\pm} = \tau \pm \theta \rightarrow -i(\tau \pm i\theta)$

$w = \tau - i\theta$



2.) Conformal transformation to complex plane

$$z = e^{w} = e^{\tau - i\sigma}$$
$$\bar{z} = e^{\bar{w}} = e^{\tau + i\sigma}$$



$\rightsquigarrow$ Radial ordering

"later"

$$R(\phi_1(z)\ \phi_2(w)) = \begin{cases} \phi_1(z)\ \phi_2(w) & \text{for } |z|>|w| \\ \phi_2(w)\ \phi_1(z) & \text{for } |w|>|z| \end{cases}$$

Advantage: conformal transformations are $\boxed{z \to z' = f(z)}$

9.2. Primary fields

transform as tensors under conformal transformations

$$\phi(z,\bar{z}) \to \phi'(z',\bar{z}') = \left(\frac{\partial z'}{\partial z}\right)^{-h} \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z,\bar{z})$$

any field on the world sheet

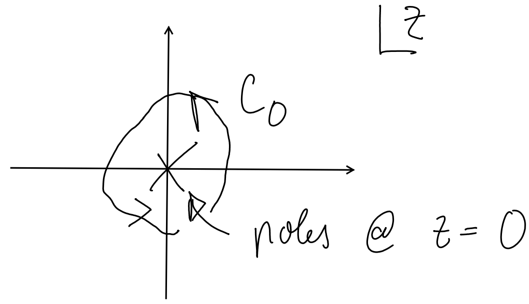
$h / \bar{h}$ conformal weights of ϕ under analytic / anti-analytic transformations

all other fields are called secondary fields

Test: $z = e^{w}$ (cylinder $\to$ plane)

$$\phi_{\text{plane}}(z) = (z)^{-h}\ \phi_{\text{cyl.}}(w)$$
$$\underbrace{\phi_{\text{plane}}}_{\text{dropped from now on}}(z) = \sum_n \phi_n\, z^{-n-h} \quad \text{(mode expansion)}$$

$$\phi_n = \oint_{C_0} \frac{dz}{2\pi i}\ \phi(z)\ z^{n+h-1}$$



9.3. Energy momentum tensor

$T_{+-} = 0 \rightsquigarrow T_{z\bar{z}} = 0$, remember due to conformal invar. ↗ EX 3.2.

conservation $\quad \partial_\alpha T^{\alpha\beta} = 0 \quad \Leftrightarrow \quad \begin{cases} \partial_{\bar z} T_{zz} + \partial_z T_{\bar z z} = 0 \\ \partial_z T_{\bar z \bar z} + \partial_{\bar z} T_{z \bar z} = 0 \end{cases}$

$\partial_{\bar z} T_{zz} = 0$

$\partial_z T_{\bar z \bar z} = 0$

$T(z) := T_{zz}(z)$ chiral and

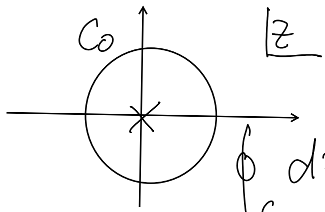
$\bar T(\bar z) := T_{\bar z \bar z}(\bar z)$ anti-chiral

gives rise to Virasoro generators L_n ↗ EX 4.3.

not only $T(z)$ is conserved but also $\xi(z) T(z)$

$$\leadsto \quad T_\xi = \oint_{C_0} \frac{dz}{2\pi i}\, \xi(z)\, T(z)$$

≙ conserved charge for conformal symmetry

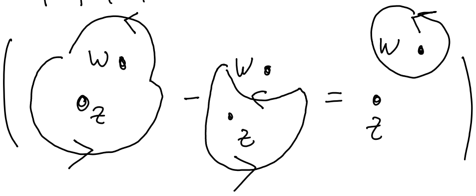


$$\oint_{C_0} dz \mathrel{\hat=} \int_0^{2\pi} d\theta \ \ldots$$

$$\delta_\xi \phi(w) = -[T_\xi, \phi(w)] \quad \hat= \text{ infinitesimal conformal transformation}$$

$$\delta_\xi \phi(w) = -\oint_{\substack{C_0 \\ |z|>|w|}} \frac{dz}{2\pi i}\, \xi(z) T(z) \phi(w) + \oint_{\substack{C_0 \\ |z|<|w|}} \frac{dz}{2\pi i}\, \xi(z) T(z) \phi(w)$$

$$= -\oint_{C_w} \frac{dz}{2\pi i}\, \xi(z) T(z) \phi(w)$$



compare with infinitesimal version for primary ϕ

$$\delta_\xi \phi(z,\bar z) = -\left(h\, \partial\xi + \xi\, \partial + \ \bar{\ } \ \right) \phi(z,\bar z)$$

and $\displaystyle \oint_{C_z} \frac{dw}{2\pi i} \frac{f(w)}{(w-z)^n} = \frac{1}{(n-1)!} f^{(n-1)}(z)$ Cauchy-Riemann formula

$$\Rightarrow \quad \boxed{T(z) \cdot \phi(w) = \frac{h\, \phi(w)}{(z-w)^2} + \frac{\partial \phi(w)}{z-w} + \text{finite terms}}$$

Operator Product Expansion OPE

9.4. Operator Product Expansion

Idea: Formalise! $\{O_i\}$ complete set of local operator

$$O_i(z,\bar{z})\, O_j(w,\bar{w}) = \sum_k C_{ij}{}^k (z-w)\, O_k(w,\bar{w})$$

example: $$\phi(z)\cdot \mathbb{1} = \sum_{n=0}^{\infty} \frac{(z-w)^n}{n!} \left(\frac{\partial}{\partial w}\right)^n \phi(w)$$

$n=0$ $\phi(w)$ primary

$n>1$ $\partial^n \phi(w)$ decendants

Covariance under rescaling requires

$$C_{ij}{}^k(z-w) = (z-w)^{h_k-h_i-h_j}\,(\bar{z}-\bar{w})^{\bar{h}_k-\bar{h}_i-\bar{h}_j}\, C_{ij}{}^k$$

just numbers, define the CFT

more complicated:

$$\boxed{T(z)\,T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \text{finite}}$$

not a primary field!

from: $T(z) = \sum_n z^{-n-2} L_n$ or $L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z)$

$$[L_m, L_n] = \oint_{C_0} \frac{dw}{2\pi i} \oint_{C_w} \frac{dz}{2\pi i}\, z^{m+1} w^{n+1} \left[\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}\right]$$

$$= \frac{c}{12} m(m^2-1)\,\delta_{n+m,0} + (m-n) L_{m+n}$$

see EX 6.