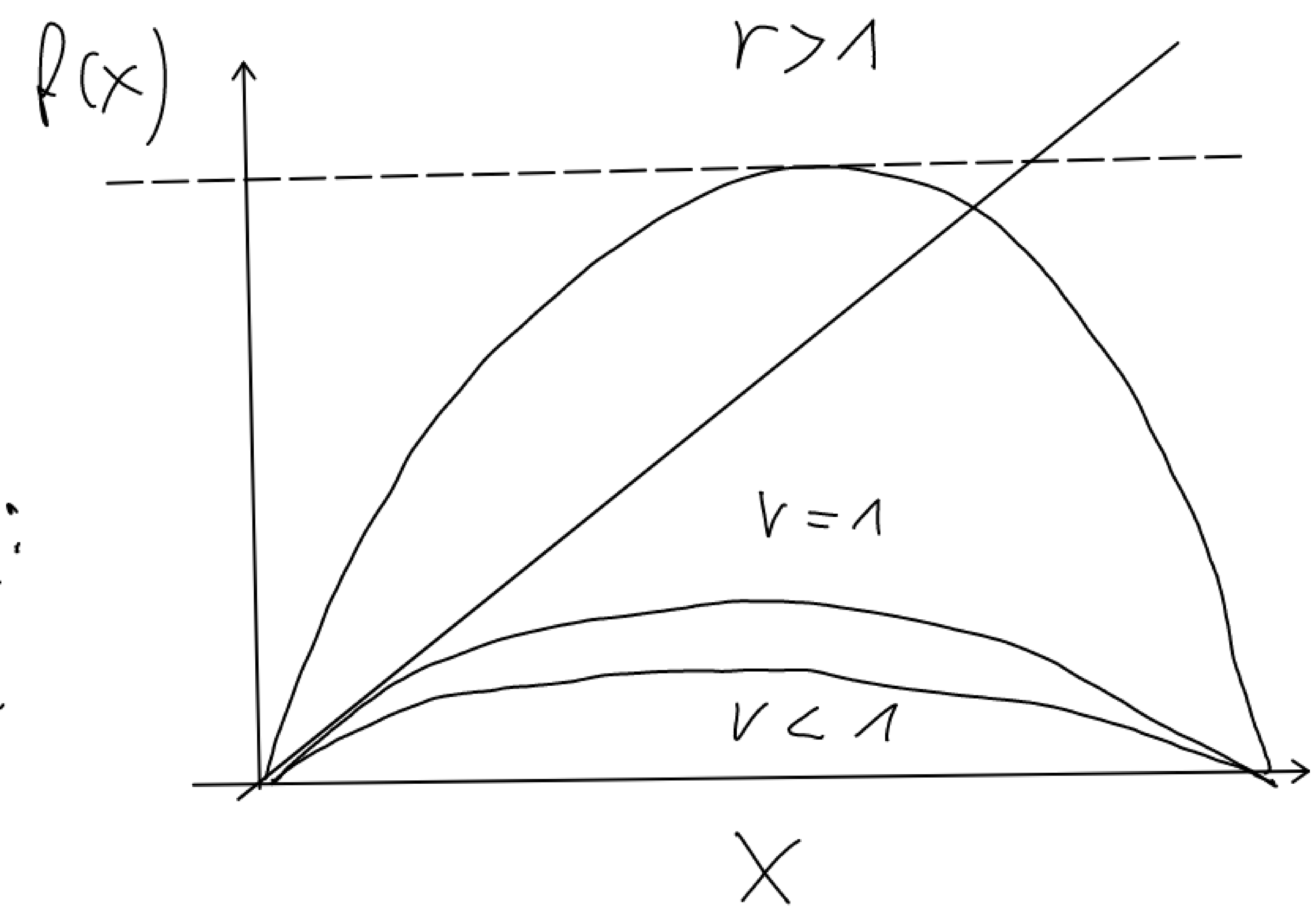
7. Iteration, bifurcation and chaos

Chaos already in 5th lecture. But complicated setup, i.e. double pendulum, ODE, etc.

Is there an even simpler setup with similar behavior?

7.1. Logistic map



$$f(x) = r\,x(1-x)$$

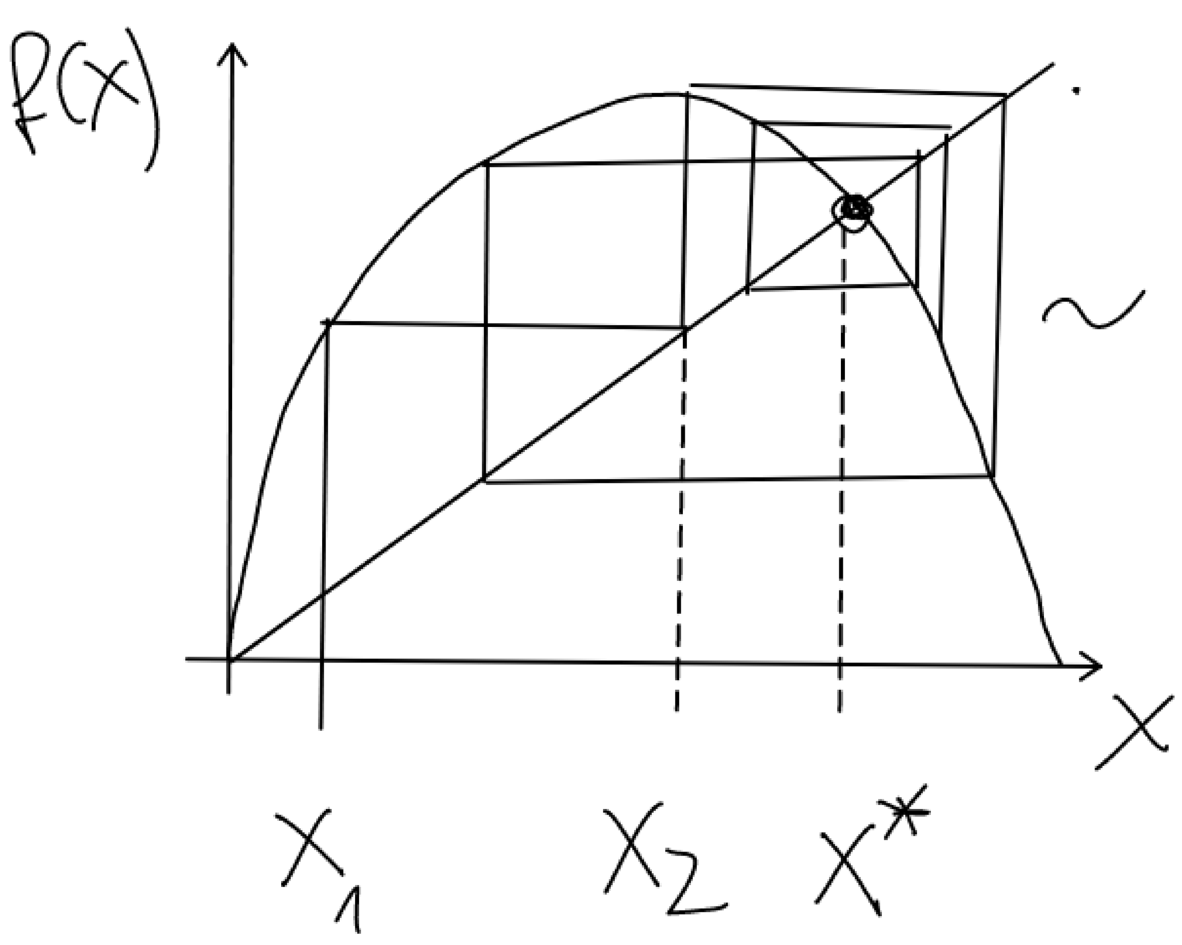
rabbit population:

$x < 1/2$ reproduction

$x > 1/2$ starvation

$r \mathrel{\hat{=}}$ rate of reproduction

Iteration $x_{i+1} = f(x_i)$ can be visualized by



~ cobweb plot

and has fixed point. When does it converge? Remember Banach's theorem.

$0 < r < 1$: (boring) $f(x)$ is contraction on $A = [0,1]$

$\Rightarrow$ All $x_0 \in A$ converge to $x^* = 0$

for $r > 1$: $x = r\,x(1-x) \Rightarrow x(\underbrace{-rx + r - 1}_{=0}) = 0$

$$x_1^* = 0 \;\&\; x_2^* = 1 - 1/r$$

If we are close to a fixed point:

$$x_{i+1} = f(x^* + \varepsilon_i) \approx \underset{\overset{\shortparallel}{x^*}}{f(x^*)} + \underbrace{f'(x^*)\,\varepsilon_i}_{\varepsilon_{i+1}}$$

$$\rightsquigarrow \quad \varepsilon_{i+1} = f'(x^*)\,\varepsilon_i \rightarrow \boxed{\begin{array}{l} |f'(x^*)| < 1 \rightarrow \text{stable (contraction)} \\ |f'(x^*)| > 1 \rightarrow \text{instable} \end{array}}$$

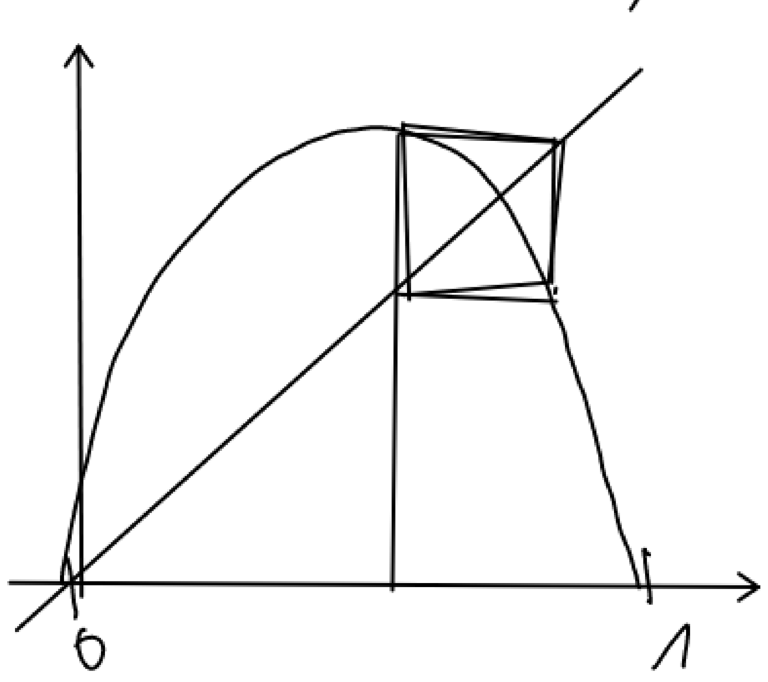
for the logistic map
$$f'(x_1^*) = r$$
$$f'(x_2^*) = 2 - r$$

$\rightsquigarrow \quad 1 < r < 3 \qquad x_1^* = 0$ instable and
$\qquad\qquad\qquad\quad x_2^* = 1 - 1/r$ stable

Observation: interation converges to stable fixed point.

3.2. Bifurcation and periodic orbits

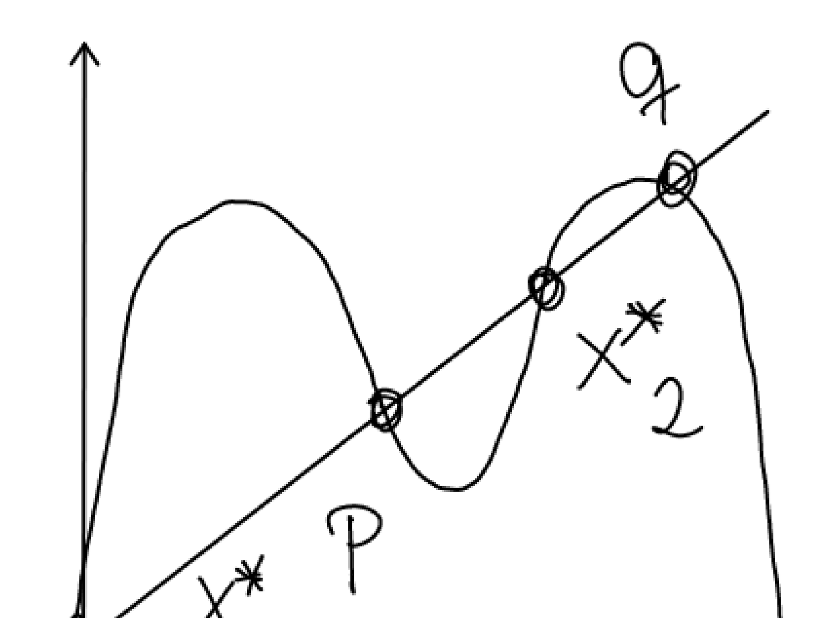
For $r = 3 + \varepsilon$, we see periodic orbits like



with period 2.

$\triangleq$ fixed point of $f^2(x) = g(x)$

two new fixed point, p & q



$x_1^* = 0, \quad x_2^* = 1 - 1/r$ (instable for $r > 3$)

$$p, q = \frac{r + 1 \pm \sqrt{(r-3)(r+1)}}{2r}$$

stable $\quad 3 < r < 3,449\ldots$

when they became instable, we go to

$h(x) = g(x)^2 = f(x)^4 \rightsquigarrow$ periodic orbit with period 4 . . .

$r <$	Period
$r_1 = 3$	2
$r_2 = 3{,}449..$	4
$r_3 = 3{,}54409..$	8
$\vdots$	$\vdots$
$r_\infty = 3{,}569946..$	∞

bifurcation

$$\lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4{,}669\ldots = \delta$$

Feigenbaum - constant

$\hookrightarrow$ new instable fixed point for $f^{2^n}(x)$ and two new stable fixed points around it.

3.3. Chaos

$r > 3{,}5699\ldots$ no periodic orbits $\rightsquigarrow$ chaos

still some structure, i.e. window with period 3

for $3{,}8284 < r < 3{,}8415 \rightsquigarrow$ check for

$r = 3{,}8282$ & $x_0 = \frac{1}{2}$

Question: How to see this structure?

Do a lot of iterations (~ 1000) starting close to $x^* = 0$, i.e. $x_0 = \varepsilon = 10^{-5}$ and plot the last 100 on the y-axis. Repeat for different r on the x-axis.

$\rightsquigarrow$ bifurcation diagram

3.4. Self-similarity and universality

"zooming" into the diagram we see the same structures

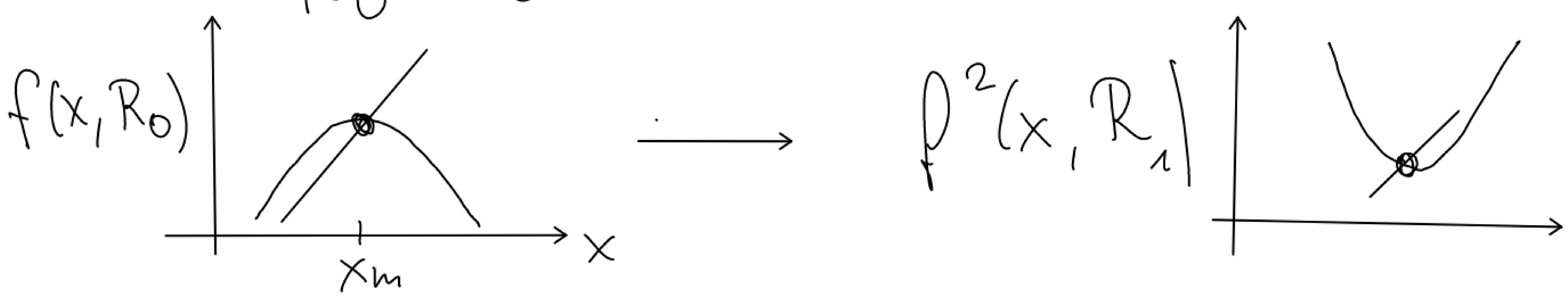
all smooth concave map with one maximum on $[0,1]$ have same δ

unimodal

Why ? RG-flow

1. superstable fixed point for $f'(x^*) = 0$
   remember from last lecture at least quadratic convergence (Newton's method with $p=2$)

2. Take $f(x)$ with maximum at x_m and choose $r = R_0$ such that $x^* = x_m$.



3. rescale the coordinates to get

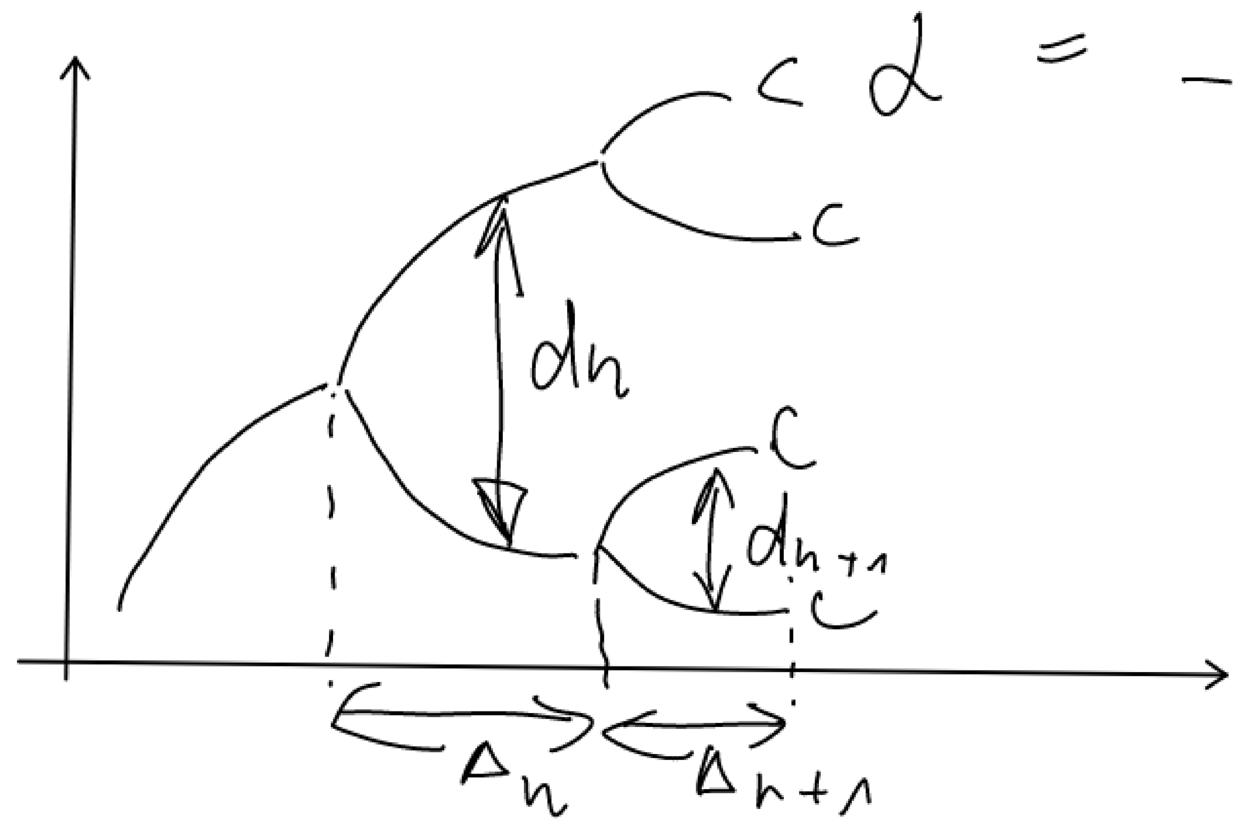
$$f(x, R_0) \simeq \alpha f^2\left(\frac{x}{\alpha}, R_1\right)$$

4. Alternatively $r = R_i$ such that x_m is on orbit of period 2^i

$\rightsquigarrow$ converges to $g_i(x) = \lim\limits_{n\to\infty} \alpha^n f^{2n}\left(\frac{x}{\alpha^n}, R_{n+i}\right)$

for $i \to \infty$ $\quad g_\infty(x) = \alpha\, g_\infty\left(\frac{x}{\alpha}\right)$ fixed point function

$\alpha = -2,5029$



$$\delta = \lim_{n\to\infty} \frac{\Delta_n}{\Delta_{n+1}}$$

$$\alpha = \lim_{n\to\infty} \frac{d_n}{d_{n+1}}$$