8. BRST Quantisation

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Motivation: get a more systematical derivation of $D=26$ (critical dim.) and $a=1$ ($::$ - const.)

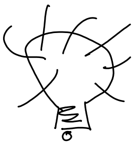
8.1. Path integral for gauge theories

Remember: In path integral formalism we eventually want to know the partition function

$$\boxed{Z=\int \mathcal{D}h\,\mathcal{D}X\, e^{i S_P[h,X]}}$$

because it is the generation function for all correlation function $\langle \dots \rangle$

$\lightning$ For gauge theories we over count because all different worldsheet metric $h_{\alpha\beta}$ are gauge equivalent.



Faddeev - Popov: Seperate gauge degrees of freedom from physical, gauge fixed ones

$\hookrightarrow$ $h_{\alpha\beta} := e^{2\phi}\hat{h}_{\alpha\beta}$ ← fixed by reparametrisations and Weyl rescalings

infinitesimal: $\delta h_{\alpha\beta} = -(P\xi)_{\alpha\beta} + 2\Lambda h_{\alpha\beta}$

Weyl rescaling

differential operator, we know $(P\xi)_{\alpha\beta} = 2\nabla_{(\alpha}\xi_{\beta)} - \nabla^\gamma \xi_\gamma h_{\alpha\beta}$

$$\mathcal{D}h = \mathcal{D}(P\xi)\mathcal{D}\Lambda = \mathcal{D}\xi\,\mathcal{D}\Lambda \cdot \left|\frac{\partial(P\xi,\Lambda)}{\partial(\xi,\Lambda)}\right|$$

$$= \left|\det\begin{pmatrix} P & 0 \\ * & 1\end{pmatrix}\right| = |\det P| = (\det P P^\dagger)^{1/2}$$

$*$ does not enter the det

$$Z=\int \mathcal{D}\xi\,\mathcal{D}\lambda \int \mathcal{D}X\,(\det P P^\dagger)^{1/2}\, e^{i S_P[e^{2\phi}\hat{h}_{\alpha\beta}, X^\mu]}$$

check later

If reparam. & Weyl symmetry not broken by quantum corrections

$$\int \mathcal{D}\xi \mathcal{D}\lambda \rightarrow \text{volume factor}$$

drops of from correlators ⇝ drop it

$$Z' = \int \mathcal{D}X^\mu \underbrace{\left(\det PP^\dagger\right)^{1/2}} e^{i S_P (e^{2\phi}\hat{h}_{\alpha\beta}, X^\mu)}$$

$$= \int \mathcal{D}c \mathcal{D}b \exp\Big(\underbrace{\tfrac{1}{2\pi}\int d^2\sigma \sqrt{-h}\, h^{\alpha\beta} b_{\beta\gamma} \nabla_\alpha c^\gamma}_{i S_{gh}[b,c]}\Big)$$

EX. 8

anti ghost $b_{\alpha\beta}$, ghost c^α

symmetric ($b_{\alpha\beta} = b_{\beta\alpha}$) and traceless ($b_\alpha^\alpha = 0$)

both Grassman odd !

⇝ break spin/statistic

8.2. Canonical quantisation of bc-ghost system

Plan: Same steps as for the X^μ

1. Mode expansion of solution to the e.o.m.

$h_{\alpha\beta}\, d\xi^\alpha d\xi^\beta = -d\xi^+ d\xi^-$ (world sheet light-cone coord.)

$$c^\pm(\sigma,\tau) = \sum_{n=-\infty}^{+\infty} c_n e^{-in(\tau \pm \sigma)}$$

$$b_{\pm\pm}(\sigma,\tau) = \sum_{n=-\infty}^{+\infty} b_n e^{-in(\tau \pm \sigma)}$$

for closed string

← drop for − —n—

2. Poisson brackets for modes

3. Quantisation

↯ results in anti-commutator

$\{b_m, c_n\} = \delta_{m+n}$, same for bared &

$\{b_m, b_n\} = \{c_m, c_n\} = 0$ $\qquad \{\circ, \bar{\circ}\} = 0$

also note that $c_n^+ = c_{-n}$ and $b_n^+ = b_{-n}$

(4.) Energy momentum tensor from S_{gh}

$$T_{\pm\pm} = -i\left[2 b_{\pm\pm} \partial_\pm c^\pm + (\partial_\pm b_{\pm\pm}) c^\pm\right]$$

with mode expansion

$$L_m^{gh} = \sum_{n=-\infty}^{+\infty} (m-n) \underbrace{: b_{m+n} c_{-n} :}$$

↯ normal ordering b_n, c_n with $n>0$ to the right

and ghost Virasoro algebra

$$[L_m^{gh}, L_n^{gh}] = (m-n) L_{m+n}^{gh} + \frac{1}{12}(-26 m^3 + 2m)\delta_{m+n}$$

8.3. Critical dim. and normal ordering const.

everything together: $L_m = L_m^X + L_m^{gh} - a\,\delta_m$

with $[L_m, L_n] = (m-n) L_{m+n} + A(m)\,\delta_{m+n}$ absent in Witt alg.

and $A(m) = \frac{D}{12} m (m^2 - 1) + \frac{1}{6}(m - 13 m^3) + 2am$

$\overset{!}{=} 0$ to cancel Weyl anomaly

$\Rightarrow$ $\boxed{D = 26 \quad \text{and} \quad a = 1}$

8.4. Physical states & BRST cohomology

Idea: Theory with local gauge symmetry generated by

$$[K_i, K_j] = f_{ij}{}^k K_k \qquad i,j,k = 1, \dots, \dim G$$

structure constants — Lie Group

Now define $\boxed{Q := c^i \left(K_i - \frac{1}{2} f_{ij}{}^k c^j b_k\right)}$

with ghosts c^i and anti-ghosts b_i governed by

$$\{c^i, b_j\} = \delta^i_j$$

and properties:

① Q is nilpotent,

Jacobi identity

$$Q^2 = \tfrac{1}{4} f_{[ij}{}^{k} f_{l]k}{}^{m} c^j c^i c^l b_m = 0$$

② BRST transformations

$$\delta c^i = \{Q, c^i\} = -\tfrac{1}{2} f_{kl}{}^{i} c^k c^l$$

$$\delta b_i = \{Q, b_i\} = K_i - f_{ij}{}^{k} c^j b_k := \tilde{K}_i$$

③ $\tilde{K}_i$ generate G, $[\tilde{K}_i, \tilde{K}_j] = f_{ij}{}^{k} \tilde{K}_k$

Ghost number $\quad N_{gh} = -\sum_{i=1}^{\dim G} b_i c^i$

with $[N_{gh}, c^i] = c^i$, $[N_{gh}, b_i] = - b_i$ and $[N_{gh}, Q] = Q$

explains name anti-ghost

Cohomology (math): $Q|\phi\rangle = 0$

called closed

trivial example: $|\phi\rangle = Q|\lambda\rangle$

called exact $\quad Q|\phi\rangle = Q^2|\lambda\rangle = 0$ ✓

$$H^n = \frac{\{|\phi\rangle \in \mathcal{H} \mid Q|\phi\rangle = 0,\ N_{gh}|\phi\rangle = n|\phi\rangle\}}{\{|\phi\rangle \in \mathcal{H} \mid \exists |\lambda\rangle \text{ with } |\phi\rangle = Q|\lambda\rangle, \text{ -n- }\}}$$

$$\boxed{= \frac{\text{closed}}{\text{exact}} \text{ with ghost number } n}$$