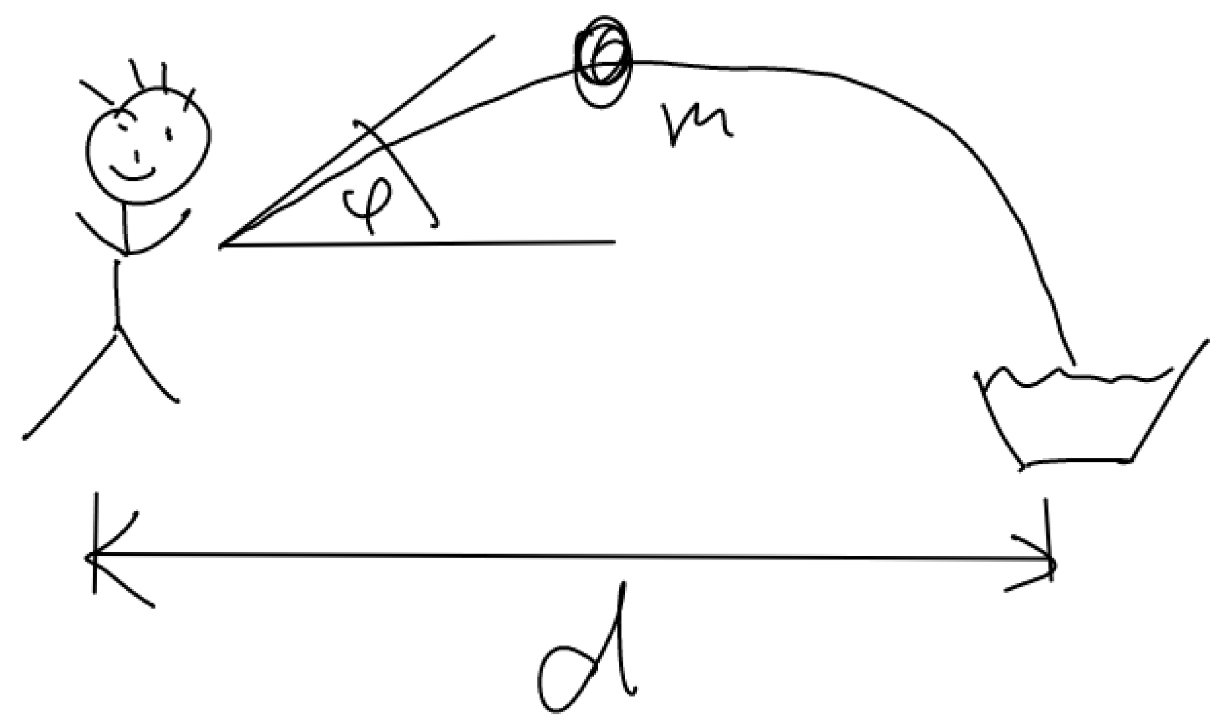
6. Methods to solve non-linear equations

Remember 4th lecture. Question: What angle we have to use to hit the bucket?



We know $\quad X(t) = V_x \cdot t \quad$ and

$$V_x = V_0 \cdot \cos\varphi$$

moreover, $\quad y(t) = y_0 + V_{y_0} t - \frac{1}{2} g \cdot t^2$

$V_{y_0} = V_0 \cdot \sin\varphi$

positive answer

$\rightsquigarrow \; y(t) = 0 \; \rightarrow \; t = -\frac{1}{g}\left[-V_0 \sin\varphi \pm \sqrt{V_0^2 \sin^2\varphi + 2 y_0 g}\right]$

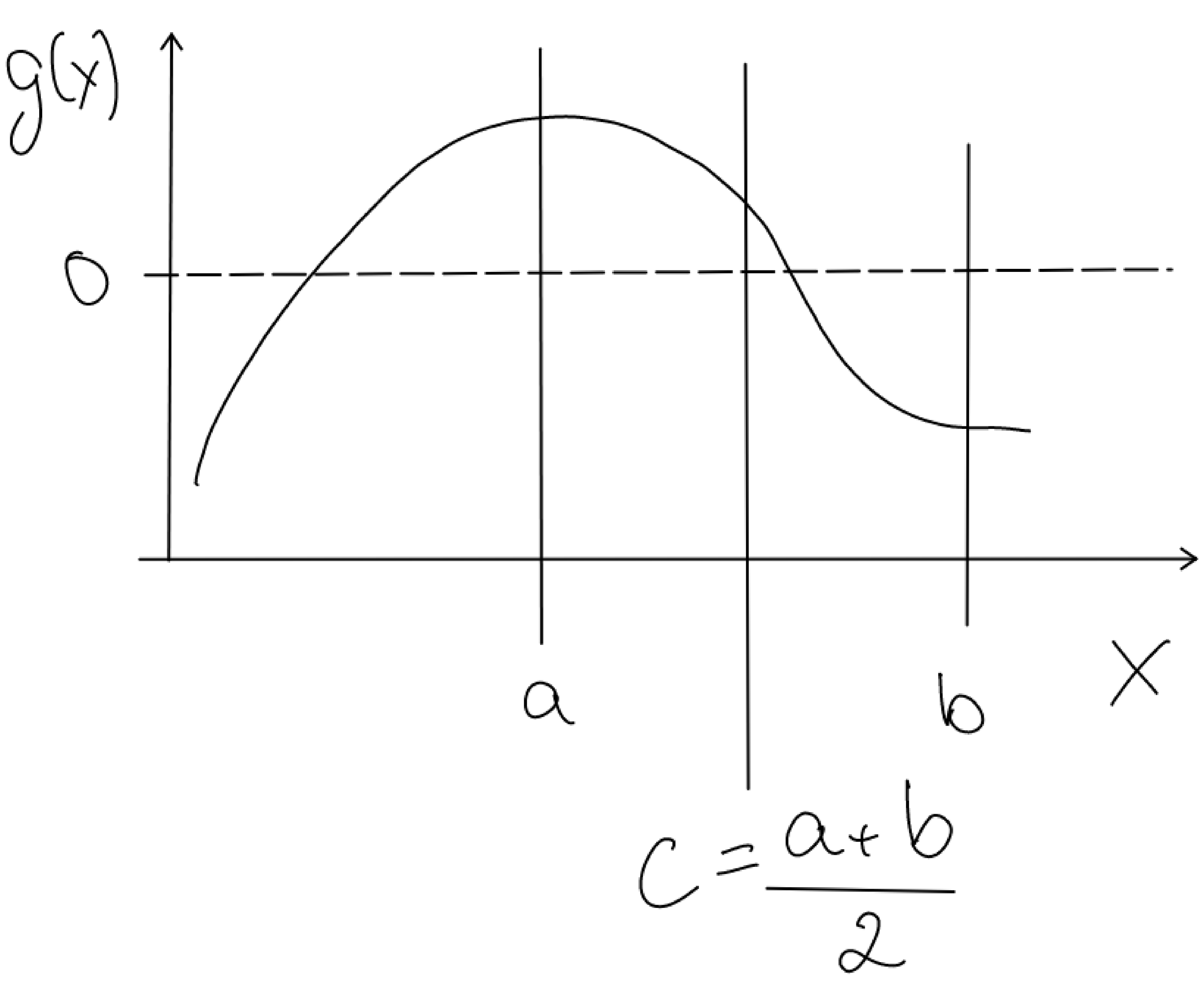
$$\boxed{d(\varphi) = x(t) = \frac{V_0 \cos\varphi}{g}\left[V_0 \sin\varphi + \sqrt{V_0^2 \sin^2\varphi + 2 y_0 g}\right]}$$

We need the inverse function!

In general $\quad f(x) = C \iff g(x) = f(x) - C = 0$

$\rightsquigarrow$ How to find roots of functions?

6.1. Bisection method



Choose interval $[a, b]$ such that $g(a) \gtrless 0$ and $g(b) \lessgtr 0$

If $g(x)$ continuous at least one root in this interval

Split the initial interval in the middle and check $\quad g(c) = 0 \quad$ done!

$\text{sign}[g(c)] = \text{sign}[g(a)]$

same $\rightarrow$ $a = c$, $b = b$

different $\rightarrow$ $a = a$, $b = c$

and repeat until $b-a$ becomes small enough.

Example: $g(x) = e^x \log x - x^2 = 0$

in interval $[1, 2] \rightsquigarrow x = 1{,}6946\ldots$

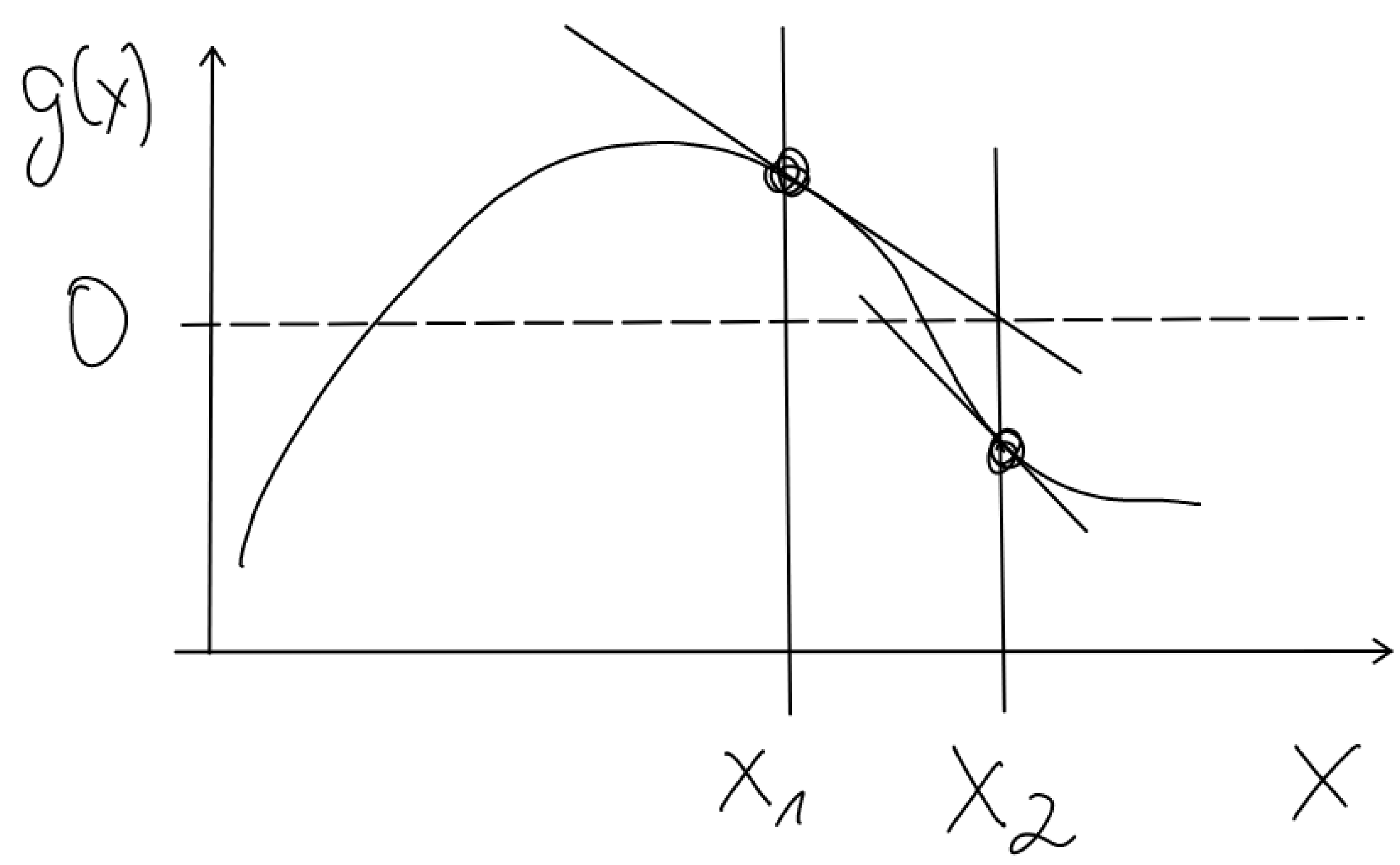
Remark: works with any function, also those which can be obtained only numerically

we need many steps. Is there something faster?

6.2. Newton's method

Take into account the derivative of the function!

we learned how to get it in 3rd lecture



$$g(x_1) + g'(x_1)\cdot(x_2 - x_1) = 0$$

$$\rightsquigarrow x_2 = x_1 - \frac{g(x_1)}{g'(x_1)}$$

and repeat

$$\boxed{x_{i+1} = x_i - \frac{g(x_i)}{g'(x_i)}} \longrightarrow \text{boils down to iteration}$$

$$x_{i+1} = f(x_i)$$

Question: When does this converge?

We are looking for a fixed point $\boxed{x^* = f(x^*)}$

norm

If $|f(x) - f(y)| < L|x-y|$ holds for a fixed constant L, the map f is called <u>Lipschitz continuous</u>. If $L<1$ this map is a <u>contraction</u>. In this case for $f: A \to A$

Banach fixed-point theorem

(i) f has exactly one fixed point x^*

(ii) for any initial value $x_0 \in A$ $x_{i+1} = f(x_i)$ converges to $x^* \in A$

(iii) $|x_n - x^*| \leq \frac{L^n}{1-L} |x_1 - x_0|$

"speed" of convergence from convergence order p

$$\limsup_{n\to\infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|^p} = K < \infty$$

for Newton's method: $f(x) = x - \frac{g(x)}{g'(x)}$, $f'(x) = \frac{g(x)g''(x)}{g'(x)^2}$

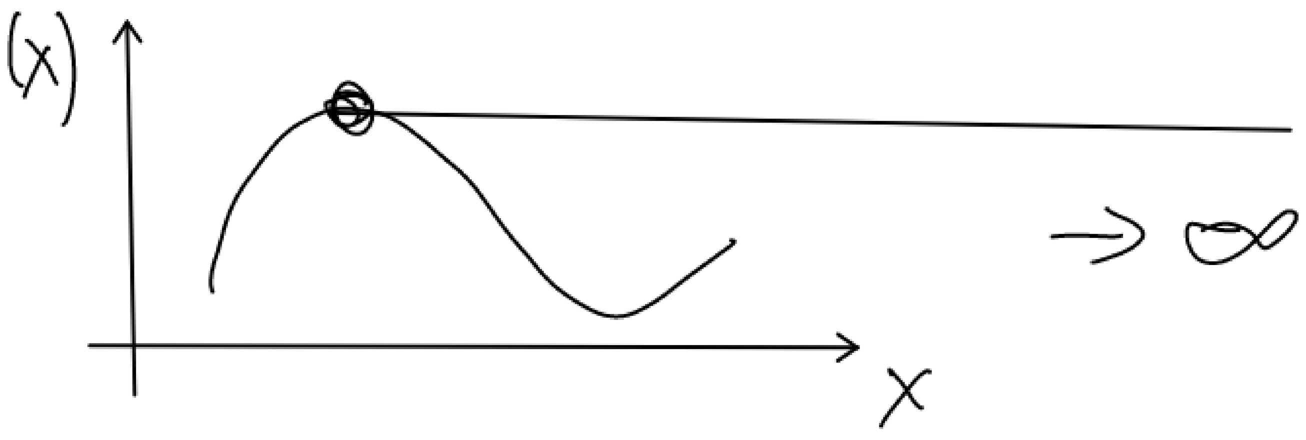
$\Delta x_n = |x_n - x^*|$

$$\begin{aligned} |x_{n+1} - x^*| &= |f(x_n) - f(x^*)| \\ &= |f'(x_n)|\Delta x_n + \mathcal{O}(\Delta x_n^2) \end{aligned}$$

$f'(x_n) = 0$ close to x^* because $g(x^*) = 0$

$\rightsquigarrow$ quadratic convergence ($p=2$)

↯ Initial value has to be chosen carefully. Avoid $g'(x_0) \approx 0$.



For higher dimensions: Jacobi matrix

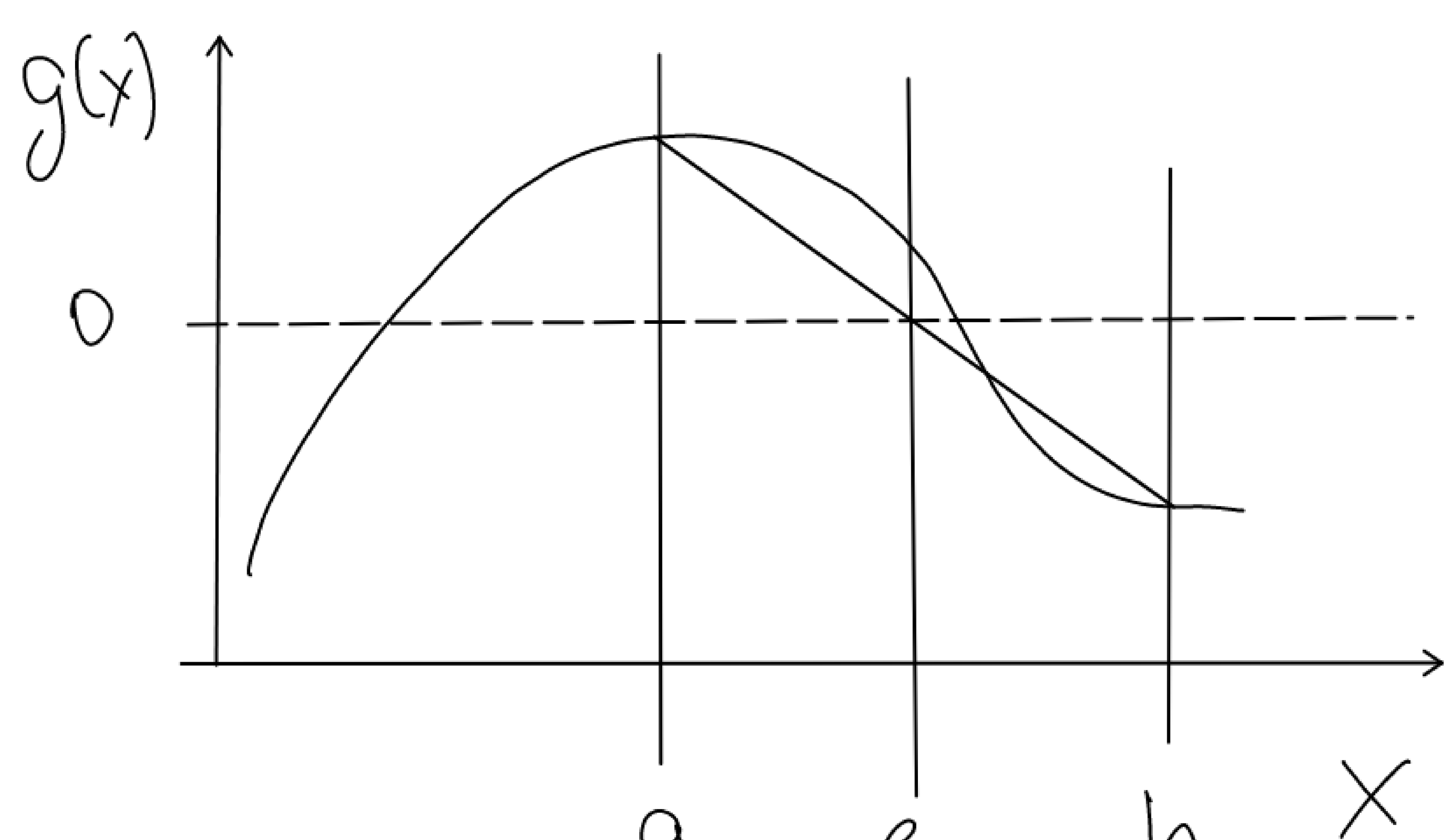
$$J_{ij}(\vec{x}) = \frac{\partial g_i}{\partial x^j}\Big|_{\vec{x}} \quad \text{and} \quad \boxed{\vec{x}_{i+1} = \vec{x}_i - J^{-1}(\vec{x}_i) \cdot \vec{g}(\vec{x}_i)}$$

Note: There is no bisection method in dim > 1.

6.3. Secant method

Computing the derivative might be a problem.

Hyprid between Newton's and bisection method.



$$c = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

then follow the same steps as in 6.1.

usually converges with $p = \frac{1+\sqrt{5}}{2}$ (golden ratio)

$\longrightarrow$ better than bisection with $p = 1$