last time started to implement constraints to identify $\boxed{\text{"physical states"}}$ $\rightsquigarrow$ Question: What is their mass?

remember: $L_0 = \underbrace{\frac{1}{2}\alpha_0^2}_{-A^2/2} + \underbrace{\sum_{p=1}^{\infty} \alpha_{-p}\cdot\alpha_p}_{N \,\hat{=}\, \text{level}}$

$(H = L_0 + \bar{L}_0)$

$$\text{mass} = M^2 = \begin{cases} 2/\alpha' \, A^2 & \text{closed string} \\ 1/2\alpha' \, A^2 & \text{open string} \end{cases}$$

and $(L_0 - a)|\Psi\rangle = 0$, $(\bar{L}_0 - \bar{a})|\Psi\rangle = 0$

$\rightsquigarrow$ $A^2 = N - a = \bar{N} - \bar{a}$ with $a = \bar{a}$ & $N = \bar{N}$ level matching

States: Level 0: $|\Psi\rangle = |p\rangle$

$A^2 = -2a < 0$ for $a > 0$

$\rightsquigarrow M^2 < 0 \rightsquigarrow$ tachyon

Level 1 $|\Psi\rangle = \varepsilon_\mu \alpha_{-1}^\mu |p\rangle$ open string $\rightsquigarrow$ vector

$|\Psi'\rangle = \varepsilon_{\mu\nu} \alpha_{-1}^\mu \bar{\alpha}_{-1}^\nu |p\rangle$ closed string $\rightsquigarrow$ tensor

Wait! Check constraints!

$(L_0 - a)|\Psi\rangle = 0$ by construction

$L_1|\Psi\rangle = \varepsilon_\mu [L_1, \alpha_{-1}^\mu]|p\rangle$ because $L_1|p\rangle = 0$

$= \varepsilon_\mu \alpha_0^\mu |p\rangle \overset{!}{=} 0$

$\rightsquigarrow$ $\varepsilon_\mu p^\mu \overset{!}{=} 0 \rightsquigarrow \varepsilon \perp p$

moreover: $\langle\Psi|\Psi\rangle = \varepsilon^2 = \varepsilon^\mu \varepsilon_\mu > 0$

 use unitarity to constrain a

$\cancel{\boxed{a>1}}$ $p \in \mathbb{R}^{1,d} \longrightarrow \varepsilon \in \mathbb{R}^{1,d-1}$

$d-1$ states with positive $\langle\psi|\psi\rangle$
and one with neg. $\langle\psi|\psi\rangle$ = "ghost" :-c

$\boxed{0 \leq a < 1}$ $p^2 < 0 \longrightarrow \varepsilon \in \mathbb{R}^{d}$

d states with positive $\langle\psi|\psi\rangle$
$\rightsquigarrow$ massive vector particle in QFT Proca action

$\boxed{a=1}$ $p^2 = 0 \longrightarrow \varepsilon \in \mathbb{R}^{d}$
$(d-1)$ states with positive $\langle\psi|\psi\rangle$
1 zero norm state for $\varepsilon \sim p$
$\hat{=}$ null state
$\hookrightarrow$ remember: same for quantisation of EM field (only transversal modes)

Open strings can describe gauge theories

closed string

$L_1 |\psi'\rangle = \bar{L}_1 |\psi'\rangle = 0 \rightsquigarrow p^\mu \varepsilon_{\mu\nu} = \varepsilon_{\nu\mu} p^\mu = 0$

for $a > 1$ ghosts, for $a = 1$ null states with

$$\varepsilon_{\mu\nu} = \underbrace{p_\mu a_\nu}_{d} \text{ or } \underbrace{a_\mu p_\nu}_{d} , \quad a_\mu p^\mu = 0$$

$$d + d = 2d - 1$$

because $\langle\psi|\psi\rangle = \varepsilon_{\mu\nu}\varepsilon^{\mu\nu}$

$d^2 - 2d + 1 = (d-1)^2$ physical states

fundamental of the little group $SO(d-1)$

$$\underset{\varepsilon_\mu}{(d-1)} \otimes \underset{\nu}{(d-1)} = 1 \oplus \frac{d(d-1)}{2} - 1 \oplus \frac{(d-1)(d-2)}{2}$$

"trace" $(\varepsilon \sim \mathbb{1})$ 1 state = dilaton

symmetric, traceless = graviton

antisymmetric

↯ "String theory contains gravity"

Summary ① Fock space $\mathcal{F} \ni |n_i, p\rangle$

② $\mathcal{F}_{\text{phys}} = \{ \psi \in \mathcal{F} \mid \underbrace{L_1 \psi = 0 ,\ L_2 \psi = 0}_{}, (L_0 - a)\psi = 0 \}$

implies $L_n |\psi\rangle = 0$ for $n \geq 1$ because $[L_m, L_n] \sim L_{m+n}$

③ $\mathcal{F}_{\text{spurious}} = \{ \chi \in \mathcal{F},\ (L_0 - a)\chi = 0 \quad \langle \chi, \psi \rangle = 0 \quad \forall \psi \in \mathcal{F}_{\text{phys.}} \}$

④ $\mathcal{F}_{\text{null}} = \mathcal{F}_{\text{phys}} \cap \mathcal{F}_{\text{spurious}}$, in particular

$\psi \ni \mathcal{F}_{\text{null}} \to \langle \psi | \psi \rangle = 0$

$$\boxed{\mathcal{H}_{OCQ} = \mathcal{F}_{\text{phys}} / \mathcal{F}_{\text{null}}}$$

OCQ: old covariant quantisation

6. Critical dimension

Motivation: open string state @ level $N=2$

$$|\psi\rangle = \varepsilon_{\mu\nu}\, \alpha_{-1}^{\mu} \alpha_{-1}^{\nu} |p\rangle + \xi_\mu \alpha_{-2}^{\mu} |p\rangle$$

$\varepsilon_{\mu\nu} = \varepsilon_{\nu\mu}$

$$(L_0 - a)|\psi\rangle = 0 \quad \to \quad A^2 = 4 - 2a > 0 \qquad (0 \leq a \leq 1)$$

In rest frame $\alpha_0^0 = A$, $\alpha_0^i = 0$

$$L_1 |\psi\rangle = (\alpha_2 \alpha_{-1} + \alpha_1 \alpha_0)|\psi\rangle$$

$$= \underbrace{(2A\, \varepsilon_{0\mu} + 2\xi_\mu)}_{\overset{!}{=}0} \alpha_{-1}^{\mu} |p\rangle \qquad (1)$$

$$L_2 |\psi\rangle = \left(\alpha_2 \alpha_0 + \frac{1}{2}\alpha_1^2\right)|\psi\rangle$$

$$= \underbrace{(\varepsilon^{\mu}{}_{\mu} + 2A\,\xi_0)}_{\overset{!}{=}0} |p\rangle \qquad (2)$$

and $\langle\psi|\psi\rangle = \varepsilon_{\mu\nu}\,\varepsilon^{\mu\nu} + \xi^\mu \xi_\mu$

$$= \varepsilon_{00}^2 - 2\sum_i \varepsilon_{0i}^2 + \sum_{ij}\varepsilon_{ij}^2 - \xi_0^2 + \sum_i \xi_i^2 > 0 \quad (3)$$

traceless part of ε_{ij} is irrelevant for (1) + (2) and contributes positive to (3)

$\rightsquigarrow$ assume $\varepsilon_{ij} = \varepsilon\,\delta_{ij}$

(1) $\xi_i = -A\,\varepsilon_{0i}$

$\xi_0 = -A\,\varepsilon_{00}$

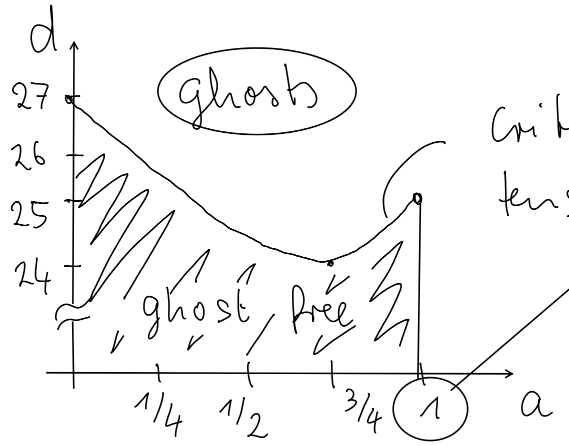
(2) $2A\xi_0 = \varepsilon_{00} - \varepsilon d$

$$\xi_0 = -\frac{Ad}{1+2A^2}\,\varepsilon$$

$$\varepsilon_{00} = \frac{d}{1+2A^2}\,\varepsilon$$

(3) becomes $d \le \dfrac{(9-4a)^2}{3-2a}$ $\qquad d = D - 1$

space spacetime



critical line for trace of rank 2 tensor

@ $a=1$, $d=25$ one (scalar) null state $|\chi\rangle$

$$|\chi\rangle = \left(5\alpha_{-1}^{0\,2} + \sum_i \alpha_{-1}^{i\,2} - 5A\,\alpha_{-2}^{0}\right)|p\rangle$$

please check! $= \left(2L_{-2} + 3L_{-1}^2\right)|p\rangle$ physical state

$$\Rightarrow \langle\chi|\psi\rangle = \langle p|\,2L_2 + 3L_1^2\,|\psi\rangle = 0$$

$\mathcal{H}_{OCQ}$ is a pre-Hilbert space (inner product is pos. def.) if and only if (1.) $a=1$ and $D=26$

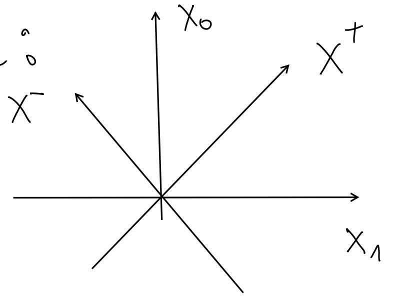
(2.) $a\le 1$ and $D \le 25$

No - ghost theorem

(1.) is called critical string (theory) with "unusual large number of null states" in $\mathcal{F}_{phys}$ of the form $\sum_{n\geq 1} c_n L_{-n} |\chi\rangle$

7. Light-cone gauge

Idea:



$$X^\mu \begin{cases} X^\pm = \frac{1}{\sqrt{2}}\left(X^0 \pm X^1\right) \\ X^I \text{ for } I = 2, \dots, d \end{cases}$$

unchanged

metric becomes $-(dX^0)^2 + \sum_i (dX^i)^2 = -2\,dX^+ dX^- + \sum_I (dX^I)^2$

i.e. $p_+ = -p^-$, $p_- = -p^+$, and $p_I = p^I$

gauge choice: $X^+(\tau, \sigma) = \begin{cases} p^+ \alpha' \tau & \text{(closed string)} \\ 2p^+ \alpha' \tau & \text{(open string)} \end{cases}$

make it easy to solve constraints, like in EX 5.1 b)

↗ see EX 7 for all details