5. Quantization of gauge theories

↯ Local symmetry results in infinitely many equivalent field configurations.

Example: the photon Propagator

$$S = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\int d^4x\, A_\mu\left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)A_\nu$$

therefore we have: $\left(\underbrace{-k^2\eta_{\mu\nu} + k_\mu k_\nu}_{}\right) G^{\nu\rho}(k) = i\delta_\mu^\rho$

↯ not invertible

Physical degrees of freedom are in the coset

$$\mathcal{A}/\mathcal{G} = \{A_\mu \sim A'_\mu : \mathcal{A},\ \exists\ U \in G \text{ with } A'_\mu = A^U_\mu\}$$

in the path integral one splits accordingly

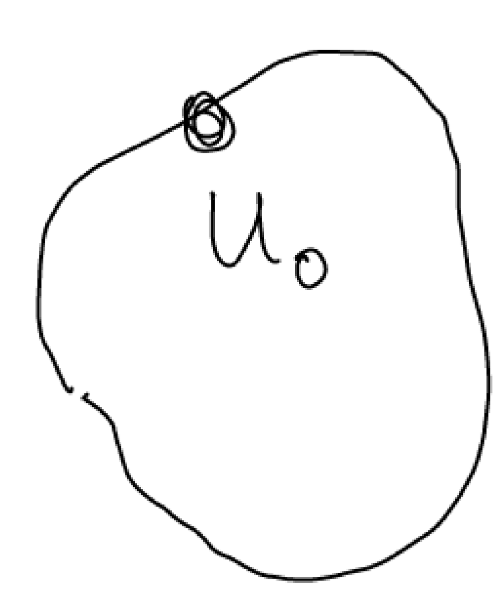
$$Z[J] = \int_{\mathcal{A}} \mathcal{D}A\, e^{iS[A] + i\int d^dx\, J^\mu A_\mu}$$

$$= \underbrace{\left(\int_{\mathcal{G}} \mathcal{D}U\right)}_{\text{global factor, can be ignored}}\left(\int_{\mathcal{A}/\mathcal{G}} \tilde{\mathcal{D}}A\, e^{iS[A] + i\int d^dx\, J^\mu A_\mu}\right)$$

$\tilde{\mathcal{D}}A$ → measure of the gauged fixed configurations

→ We define the gauge fixing function $f(A_\mu)$ such that $f(A^U_\mu) = 0$ has one unique solution U_0 for a given A_μ

$$\Delta_f[A_\mu]\int_{\mathcal{G}} \mathcal{D}U\, \delta\left(f(A^U_\mu)\right) \overset{!}{=} 1$$



gauge orbit

with the determinat

$$\Delta_f[A_\mu] = \det\left(\frac{\delta f(x)}{\delta U(y)} \Big|_{U=U_0} \right)$$

We can now rewrite

$$Z[J] = \int_A \mathcal{D}A \underbrace{\Delta_f[A_\mu] \int_\mathcal{G} \mathcal{D}U\, \delta(f(A_\mu^U))}_{1} e^{iS[A] + i\int d^dx\, J^\mu A_\mu}$$

$$\Rightarrow \int_{\mathcal{A}/\mathcal{G}} \tilde{\mathcal{D}}A = \int_\mathcal{A} \mathcal{D}A\, \Delta_f[A_\mu]\, \delta(f(A_\mu))$$

$\Delta_f[A_\mu]$ can be expressed by a path integral

$$\Delta_f[A_\mu] = \int \mathcal{D}c\, \mathcal{D}\bar{c}\; e^{iS_{gh}}$$ ghost fields

Can we do the same for $\delta(f(A_\mu))$?

generalize gauge fixing condition

$f(A_\mu(x)) = B(x)$ does not depend on A_μ, will not affect $\Delta_f[A_\mu]$

$$\text{const} = \int \mathcal{D}B \exp\left(-\frac{i}{2\xi} \int d^dx\, B^2 \right)$$

$$\hookrightarrow Z[J] = \int \mathcal{D}A_\mu\, \mathcal{D}c\, \mathcal{D}\bar{c}\, \mathcal{D}B\; e^{iS[A] + iS_{gh} + i\int d^dx\, (J^\mu A_\mu - \frac{1}{2\xi}B^2)} \cdot \delta(f(A_\mu) - B)$$

after integration out B

$$Z[J] = \int \mathcal{D}A_\mu\, \mathcal{D}c\, \mathcal{D}\bar{c}\, \exp\left(i S_{eff}[A, c, \bar{c}] + i\int d^dx\, J^\mu A_\mu \right)$$

$$\boxed{S_{eff} = S + S_{gf} + S_{gh}} \qquad S_{gf} = -\frac{1}{2\xi} \int d^dx\, (f_a(A_\mu))^2$$

5.1. Renormalization

Consider YM with gauge group G coupled to N_f fermions transforming in the representation R

$$\beta(g) = -\frac{g^3}{16\pi^3}\left(\frac{11}{3}\,C(\text{adj}) - \frac{4}{3}N_f\,C(R)\right)$$

index of the representation

for $SU(N)$ $C(\text{adj}) = N$ and $C(\text{fund}) = 1/2$

5.2. The large N expansion

't Hooft in 1974: $SU(N)$ YM simplifies considerably for $N \to \infty$

$\Rightarrow$ $N \to \infty$, $N_f = 0$ $\qquad \beta \to \infty$

but keeping $\lambda = g^2 \cdot N$ fixed $\quad (N \to \infty \;\&\; g \to 0)$

$$\boxed{\mu\frac{d\lambda}{d\mu} = -\frac{11}{24\pi^2}\lambda^2 + \mathcal{O}(\lambda^3)}$$

$\lambda =$ 't Hooft coupling

Let's look at a simpler toy model of a scalar in the fundamental rep. $\phi^i{}_j = \phi^a (T_a)^i{}_j$

To mimic YM's interactions consider

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(\partial_\mu\phi\,\partial^\mu\phi) + g\,\mathrm{Tr}(\phi^3) + g^2\,\mathrm{Tr}(\phi^4)$$

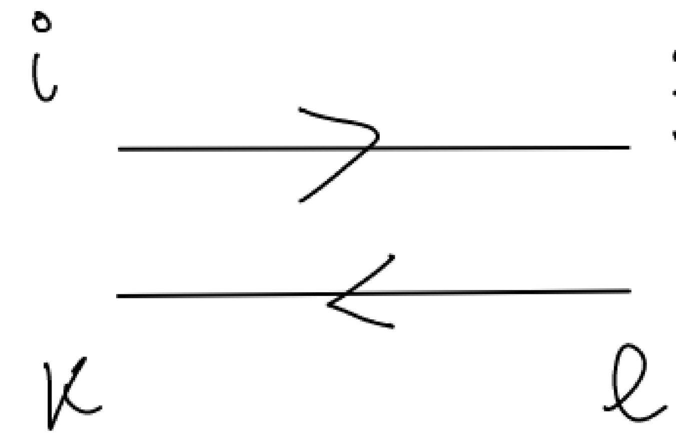
mixing $\qquad \sim g^2$

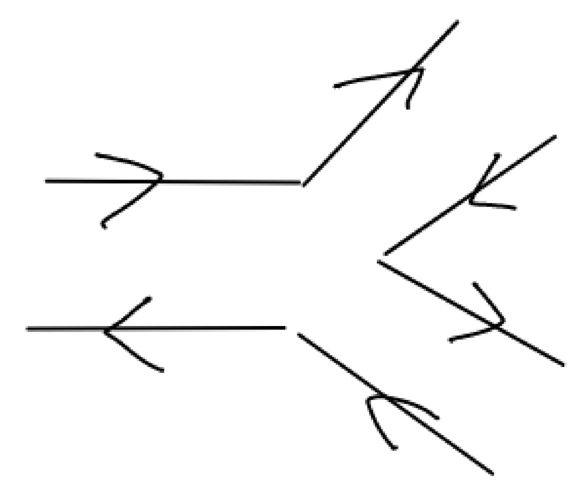
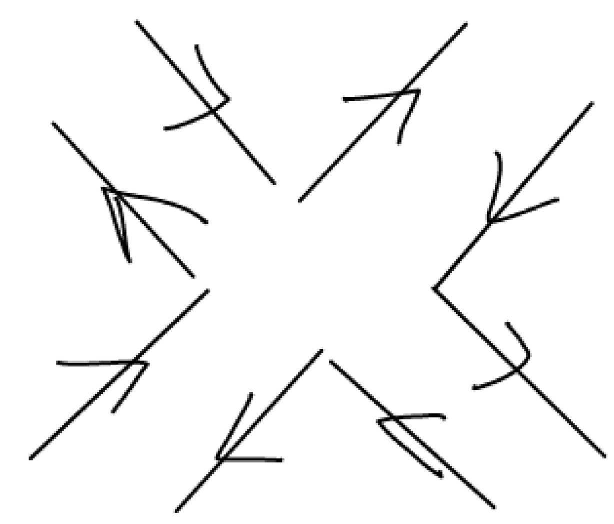
and after rescaling $\tilde{\phi} = g\,\phi$ we get

$$\mathcal{L} = \frac{1}{g^2}\left[-\frac{1}{2}\mathrm{Tr}(\partial_\mu\tilde{\phi}\,\partial^\mu\tilde{\phi}) + \mathrm{Tr}(\tilde{\phi}^3) + \mathrm{Tr}(\tilde{\phi}^4)\right]$$

for $U(N)$ we then get — from completeness $\sum_{a=1}^{N^2} (T_a)^i{}_j (T_a)^k{}_l = \delta^i{}_j \delta^k{}_l$

$$\langle \tilde{\Phi}^i{}_j(x)\, \tilde{\Phi}^k{}_l \rangle = \underbrace{\delta^i{}_j\, \delta^k{}_l}\, \frac{g^2}{4\pi^2 (x-y)^2}$$

$\hat{=}$  $\sim g^2 = \frac{\lambda}{N}$ and verteces

 $\sim g^{-2} = \frac{N}{\lambda}$  $\sim \frac{N}{\lambda}$

and for each closed loop a factor of N from trace

$\rightsquigarrow$ for a diagram with V vertices, E propagators and F loops we get $N^{V-E+F}\lambda^{E-V} = N^{\chi}\lambda^{E-V}$

where $\chi = V - E + F = 2 - 2g$ Euler characteristic, g: genus

with this the generating function W reads

$$W = \ln Z = \sum_{g=0}^{\infty} N^{2-2g} \sum_{i=0}^{\infty} \lambda^i c_{g,i}$$

 Same form as for closed string perturbation theory $\rightsquigarrow$ important hint towards the AdS/CFT correspondence.