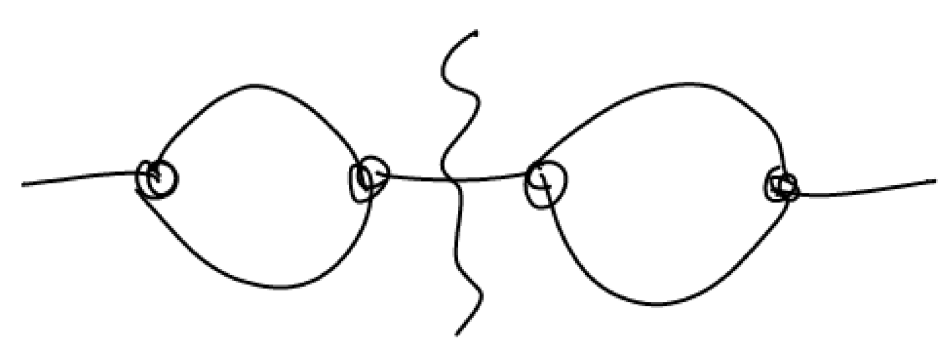
last lecture QFT intro with Most Valuable Player generating function $Z[J]$ but hard to get.

4.3. Further generating functionals

$$\boxed{Z[J] = e^{iW[J]}}$$ where $W[J]$ generates connected n-point functions, i.e.

$$\langle \phi(x)\,\phi(y)\rangle_c = \langle \phi(x)\,\phi(y)\rangle - \langle\phi(x)\rangle\langle\phi(y)\rangle, \text{ with}$$

$$\langle \phi(x_1)\ldots\phi(x_n)\rangle_c = (-i)^{n-1}\,\frac{\delta^n W[J]}{\delta J(x_1)\ldots\delta J(x_n)}\Bigg|_{J=0}$$

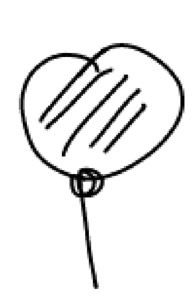
an example is  which is

one-particle reducible (cutting one propagator makes it disconnected)

the opposite are one-particle irreducible (1PI) diagrams

they are generated by (the effective action) $$\boxed{\Gamma[\varphi] = W[J] - \int d^d x\, J(x)\varphi(x)}$$

$$\varphi(x) = \langle\phi(x)\rangle_J = \frac{\delta W[J]}{\delta J(x)}$$ expectation value in the presence of sources

assume no tadpoles  then $\varphi(x)\big|_{J=0} = 0$ and we define

$$\Gamma^{(n)}(x_1,\ldots,x_n) = \frac{\delta}{\delta\varphi(x_1)}\cdots\frac{\delta}{\delta\varphi(x_n)}\,\Gamma[\varphi]$$

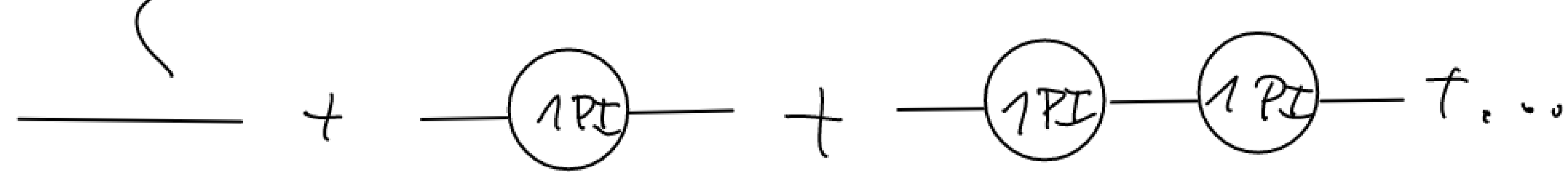
$$\langle\phi(x_1)\ldots\phi(x_n)\rangle_{1PI} = \Gamma^{(n)}(x_1,\ldots,x_n)\Big|_{J=0}$$

after Fourier transformation we find

$$\Gamma[\varphi] = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\, \varphi(-p)\left(p^2+m^2-\Pi(p^2)\right)\varphi(p)$$
$$\sum_{n=3}^{\infty} \frac{1}{n!}\int \frac{d^d p_1}{(2\pi)^d}\cdots\int\frac{d^d p_n}{(2\pi)^d}\,(2\pi)^d\,\delta(p_1+\dots+p_n)\,\cdot$$
$$\Gamma^{(n)}(p_1,\dots,p_n)\,\varphi(p_1)\dots\varphi(p_n)$$

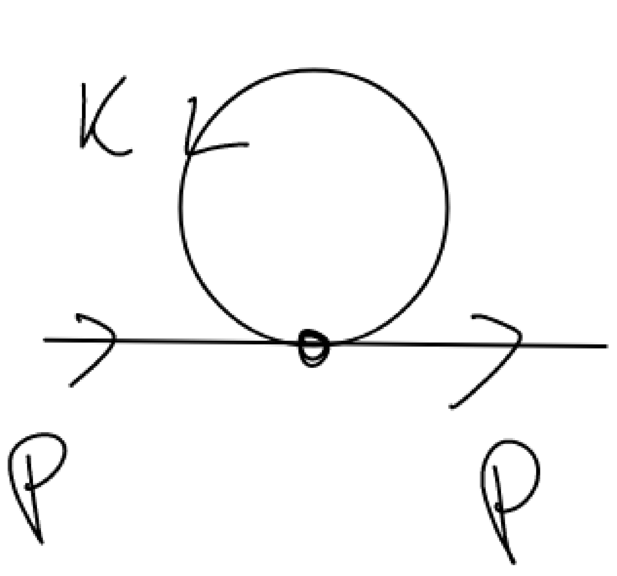
This is the classical action for the expectation value $\langle\phi(x)\rangle$. Parameters like the mass m, receive quantum corrections due to

$$\Gamma^{(2)} = \frac{1}{p^2+m^2-\Pi(p^2)}$$ self energy



4.4. Regularization and renormalization

We can now compute the one-loop contribution



to $\Gamma^{(2)}$ for ϕ^4-theory giving

$$\Gamma^{(2)}_{1L}(p,-p) = \frac{ig}{2}\int\frac{d^d k}{(2\pi)^d}\,\frac{1}{k^2+m^2-i\varepsilon}$$

after Wick rotation, $k_0 \to i k_0$, and $\varepsilon \to 0$

$$\Gamma^{(2)}_{1L}(p,-p) = -\frac{g}{2}\,\frac{\Gamma\left(1-\frac{d}{2}\right)}{(4\pi)^{d/2}}\, m^{(d-2)/2}$$

diverges for $d=4$, therefore regualarize by dimensional regularization

$$\boxed{d = 4-\varepsilon} \rightsquigarrow \Gamma^{(2)}_{1L}(p,-p) \sim \frac{1}{2}\,\frac{g}{16\pi^2}\, m^2\left(\frac{2}{\varepsilon}+1-\ln m^2\right)\left(e^{-\gamma}4\pi\right)^{\varepsilon/2}$$

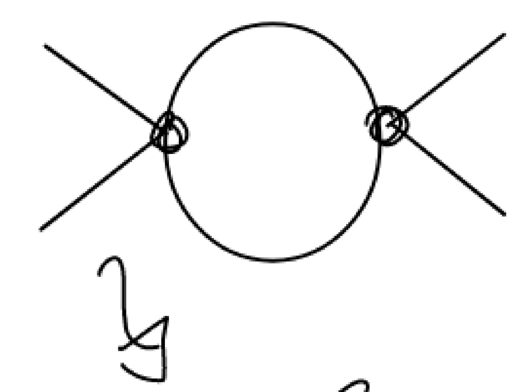
divergent part

We have quantified the divergence. No we need to remove it.

add counter terms to Lagrangian, i.e. for ϕ^4

$$\mathcal{L}_{CT} = -\frac{A}{2}\partial^\mu\phi\,\partial_\mu\phi - \frac{B}{2}\phi^2 - \frac{C}{4!}\phi^4$$

to have additional contributions



$$\Gamma^{(2)}_{CT,OL}(p,-p) = -Ap^2 - B$$

now $\quad A = 0 \quad$ and $\quad B = \frac{g m^2}{16\pi^2 \varepsilon} \quad , \quad C = \frac{3g^2}{16\pi^2\varepsilon}$

$$\Rightarrow \mathcal{L}_{bare} = \mathcal{L} + \mathcal{L}_{CT} = -\frac{1}{2}(\partial\phi_0)^2 - \frac{1}{2}m_0^2\phi_0^2 - \frac{1}{4!}g_0\phi_0^4$$

bare field and couplings

$$\phi_0 = Z_\phi^{1/2}\phi, \quad Z_\phi = 1 + A, \quad m_0 = \frac{m^2 + B}{Z_\phi}, \quad g_0 = \frac{g + C}{Z_\phi^2}$$

with the results for A, B & C from above

$$Z_\phi = 1 + \mathcal{O}(g^2), \quad m_0 = m^2\left(1 + \frac{g}{16\pi^2\varepsilon}\right), \quad g_0 = g\left(1 + \frac{3g}{16\pi^2\varepsilon}\right)$$

(minimal substraction scheme [MS])

The coupling g is for $d = 4$. But now we are in $d = 4 - \varepsilon$ $\rightsquigarrow$ $g \to g \cdot \mu^\varepsilon$

arbitrary mass scale

and same for C, but not for g_0

$g \to g + g \cdot \log\mu\, \varepsilon + \dots$ and therefore

$$\underset{\uparrow\ \text{fixed}}{m_0} = m^2\left(1 + \frac{g\log\mu}{16\pi^2} + \frac{g}{16\pi^2\varepsilon} + \mathcal{O}(\varepsilon)\right)$$

$\rightsquigarrow$ m depends on scale μ, same for g

For the Green function, we therefore have:

$$\underset{\uparrow\ \text{bare}}{G_0^{(n)}}(p_1 \dots p_n) = \langle \phi_0(p_1) \dots \phi_0(p_n)\rangle = Z_\phi^{n/2}\langle \phi(p_1)\dots\phi(p_n)\rangle$$

$$= Z^{n/2}\, \underbrace{G^{(n)}(p_1, \dots p_n)}_{\text{renormalized}}$$

↳ for the effective action

$$\Gamma_0^{(n)}(p_1, \dots, p_n) = Z_\phi^{-n/2}\, \Gamma^{(n)}(p_1, \dots, p_n)$$

$$\mu \frac{d}{d\mu} \Gamma_0^{(n)}(p_1, \dots, p_n) = 0 = \mu \frac{d}{d\mu}\left(Z_\phi^{-n/2}\, \Gamma^{(n)}(p_1, \dots, p_n)\right)$$

$\rightsquigarrow$ after chain rule

$$\boxed{\left(\mu \frac{\partial}{\partial\mu} + \beta \frac{\partial}{\partial g} + m\gamma_m \frac{\partial}{\partial m} - n\gamma\right)\Gamma^{(n)}(p_1, \dots, p_n) = 0}$$

with $\beta = \mu \frac{\partial g}{\partial \mu}$, $\gamma_m = \frac{\mu}{m}\frac{\partial m}{\partial\mu}$ and $\gamma = \frac{\mu}{2Z_\phi}\frac{\partial Z_\phi}{\partial\mu}$

β- function of coupling   anomalous dimension of ϕ

from $g_0 = g\left(1 + \frac{3g\log\mu}{16\pi^2} + \dots\right)$ we get

$$\beta = \frac{3g}{16\pi^2}$$

$$\left.\begin{array}{ll} \beta > 0 & \text{relevant} \\ \beta < 0 & \text{irrelevant} \\ \beta = 0 & \text{marginal} \end{array}\right\} \begin{array}{l} \text{operator (corresponding to the coupling)} \\ \text{for the IR} \end{array}$$