5.5. Witt algebra

remember $L_{\xi_n} = 2T \int_0^{2\pi} d\sigma\, \xi_n^+(\sigma^+)\, T_{++}$

$T_{++} = \partial_+ X \partial_+ X$

$\xi^+(\sigma^+) = e^{in\sigma^+}$

$$L_n = \frac{1}{2}\sum_p \alpha_p \alpha_{n-p}\, e^{in\tau}$$

conserved:

usually droped, but then not conserved

$$\dot{L}_n = \frac{1}{2}\sum_p i\,(\underbrace{\cancel{-p}-\cancel{n}+\cancel{p}+\cancel{n}}_{=0})\,\alpha_p\alpha_{n-p}\,e^{in\tau}$$

because $\dot{\alpha}_n = -in\alpha_n = \{H, \alpha_n\}$

again, one recovers: $\{L_n, L_m\} = i(n-m)L_{n+m}$

and $\boxed{H = L_0 + \tilde{L}_0}$

5.6. Quantisation (finally :-))

$\{\alpha_n, \alpha_m\} = in\,\delta_{m+n} \longrightarrow [\hat{\alpha}_n, \hat{\alpha}_m] = n\,\delta_{n+m}$

operators

compare with HO

$\{\alpha, \bar{\alpha}\} = i\omega \longrightarrow [\alpha, \alpha^+] = \omega$

$\alpha \sim \sqrt{\frac{\omega}{2}}(p - ix)$

ω frequency

we drop the hats

creation operator

annihilation op.

Hilbert space: $\mathcal{H}_{HO} = \overline{\mathrm{Span}\left(|\lambda\rangle \sim (a^+)^\lambda|0\rangle,\ \lambda = 0, 1, \dots\right)}$

where $a|0\rangle = 0$ and $\langle 0|0\rangle = 1$

collection of HO's of frequency n and

$$\alpha_n^\mu = \begin{cases} \text{creation operator} & n<0 \\ \text{annihilation op.} & n>0 \end{cases}$$

Remember: $\alpha_0^\mu \sim p^\mu$ ← string's center of mass momentum

$\{X_\mu, p^\nu\} = -\delta_\mu^\nu \longrightarrow [X_\mu, p^\nu] = -i\,\delta_\mu^\nu$

like for point particle, Hilbert space

$\mathcal{H}_{pp} = L^2(\mathbb{R})$ $\quad P_\mu = -i\partial_\mu \quad X^\nu = x^\nu$

plane wave $\hat{=}$ eigenstate of $\hat{P}_\mu |p\rangle = p_\mu |p\rangle$

$\langle p | p' \rangle = \delta(p_\mu - p'_\mu) \quad \langle x^\mu | p_\nu \rangle = e^{i p_\varsigma x^\varsigma} \delta^\mu{}_\nu$

String's Hilbert space: $\mathcal{H} = \mathcal{H}_{pp} \otimes \mathcal{H}_{HO's}$

$\rightsquigarrow \mathcal{H} = \mathrm{Span}_{\mathbb{C}} \{ \prod_{k>0} (\alpha_{-k})^{\lambda_k} |p\rangle, \; p_\mu \in \mathbb{R}^{D, D-1}$ and

$\lambda_k \in \{0, 1, \ldots\}$ only finitely many non-zero $\}$

with $\alpha_k |p\rangle = 0 \;\forall k>0$ and $\hat{P}_\mu |p\rangle = p_\mu |p\rangle$

Question: What happens to $L_n = 1/2 \sum_p \alpha_p \alpha_{n-p}$?

New question how to order operators ?

only annihilation & creation ops don't commute

to the right and to the left $\quad :\alpha^\mu_{-1} \alpha^\nu_1: = \alpha^\mu_{-1} \alpha^\nu_1$

$\hat{=}$ normal ordering $:\ldots: \quad :\alpha^\mu_1 \alpha^\nu_{-1}: = \alpha^\nu_{-1} \alpha^\mu_1$

$$L_n = \frac{1}{2} \sum_p :\alpha_p \alpha_{n-p}: , \quad L_0 = \frac{1}{2} \alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n} \alpha_n$$

$\hookrightarrow \langle L_n \rangle = \langle 0 | L_n | 0 \rangle = 0$

$$[L_n, \alpha^\mu_m] = 1/2 \sum_p [:\alpha_p \alpha_{n-p}:, \alpha_m]$$

$$= -m \alpha_{n+m}$$

Trick to calculate: Wick's theorem

1) $\overbracket{a\, b} = ab - :ab:$ Wick contraction

2) $:a_1 a_2: :a_3: = :a_1 a_2 a_3: + \overbracket{a_1 a_3} :a_2: + \overbracket{a_2 a_3} :a_1:$

$:a_3: :a_1 a_2: = :a_1 a_2 a_3: + \overbracket{a_3 a_1} :a_2: + \overbracket{a_3 a_2} :a_1:$

↳ $[:a_1 a_2:, :a_3:] = \left(\overbracket{a_1 a_3} - \overbracket{a_3 a_1}\right) a_2 + \left(\overbracket{a_2 a_3} - \overbracket{a_3 a_2}\right) a_1$

For the string $\overbracket{\alpha_m \alpha_n} = \alpha_m \alpha_n - :\alpha_m \alpha_n:$

$\underset{m>0}{\overbracket{\alpha_m \alpha_{-m}}} = \alpha_m \alpha_{-m} - \alpha_{-m}\alpha_m = [\alpha_m, \alpha_{-m}] = m$

$\underset{m<0}{\overbracket{\alpha_m \alpha_{-m}}} = 0$

$$= \begin{cases} m & \text{for } m>0,\ n=-m \\ 0 & \text{otherwise} \end{cases}$$

$$\overbracket{a_m a_n} - \overbracket{a_n a_m} = m\,\delta_{m+n}$$

$$[L_n, \alpha_m] = \frac{1}{2}\sum_p \Big(\underbrace{p\,\delta_{p+m}\,\alpha_{n-p}}_{p=-m} + \underbrace{(n-p)\,\delta_{n-p+m}\,\alpha_p}_{p=m+n} \Big)$$

$$= \frac{1}{2}\left(-m\,\alpha_{n+m} - m\,\alpha_{n+m}\right) = -m\,\alpha_{n+m}$$

Similar $[L_m, L_n]$ in the exercise!

1) for $n \neq m$ we find $[L_m, L_n] = (m-n) L_{m+n}$

⇝ remember $\{L_m, L_n\} = i(m-n) L_{m+n}$ ← $-i\{\cdot,\cdot\} \to [\cdot,\cdot]$

⇝ Quantisation is doing what we expect

2) ↯ $[L_m, L_{-m}] = 2m L_0 + \frac{1}{12}(m^3 - m)\cdot \underset{\text{"}c\text{"}}{1}$

central charge

one new element in the algebra

$V = \mathrm{Span}_{\mathbb{C}}\{L_n\ n \in \mathbb{Z},\ c\}$ central element $[c, L_m] = 0$

$$\boxed{[L_m, L_n] = (m-n) L_{m+n} + \frac{1}{12}(n^3-n)\, c\, \delta_{m+m}}$$

Central extension of Witt algebra = Virasoro-algebra

Remarks:

- again $SL(2,\mathbb{R})$ subalgebra by L_{-1}, L_0, L_1

  $[L_1, L_{-1}] = 2L_0 + 0 \leftarrow$ central ext. vanishes

- $[L_n, P^\mu] = 0$ and $[L_n, J^{\mu\nu}] = 0$

5.7. Physical states

Question: How to implement the constraints?

Naive from classical string: $L_n|\psi\rangle = 0 \;\; \forall n \in \mathbb{Z}$

because T_{++} vanishes under constraint

$\lightning$ Too restrictive! Problem with unitarity!

$L_n^\dagger = L_{-n}$ (unitary representation of Virasoro alge)

for every state ψ $\quad \|\psi\|^2 = \langle\psi|\psi\rangle > 0$

$$\|\underbrace{L_{-n}\psi}_{=0}\|^2 = \langle L_{-n}\psi | L_{-n}|\psi\rangle = \langle\psi| L_n L_{-n}|\psi\rangle$$

$$= \underbrace{\|L_n\psi\|^2}_{=0} + 2n\langle\underbrace{\psi|L_0|\psi}_{=0}\rangle + \frac{c}{12}(n^3-n)\langle\psi|\psi\rangle$$

$\hookrightarrow \quad \frac{c}{12}(n^3-n)\langle\psi|\psi\rangle = 0 \quad \lightning$ for $n \neq 0, \pm 1$

$\rightarrow$ only $L_n|\psi\rangle = 0$ for $\forall\, n \geq 0$

$\|L_{-1}\psi\|^2 = 2\underbrace{\langle\psi|L_0|\psi\rangle}_{=0} \quad \lightning$ conformal weights are positive

$\rightarrow \quad (L_0 - a)\,|\psi\rangle = 0$

normal ordering constant, calculated later

same for $\bar{L}_0$ and $a = \bar{a}$

$\rightsquigarrow \quad (L_0 - \bar{L}_0)\,|\psi\rangle = 0 \quad \hat{=}$ level matching