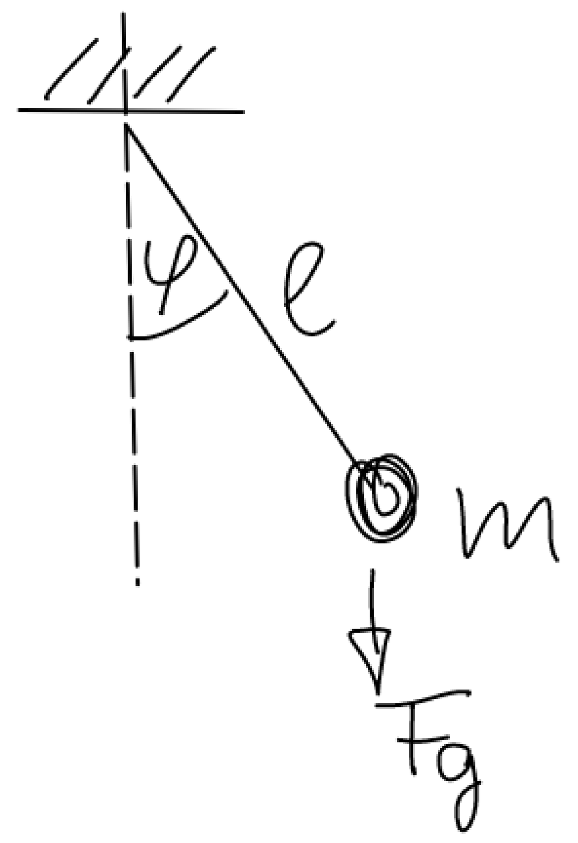
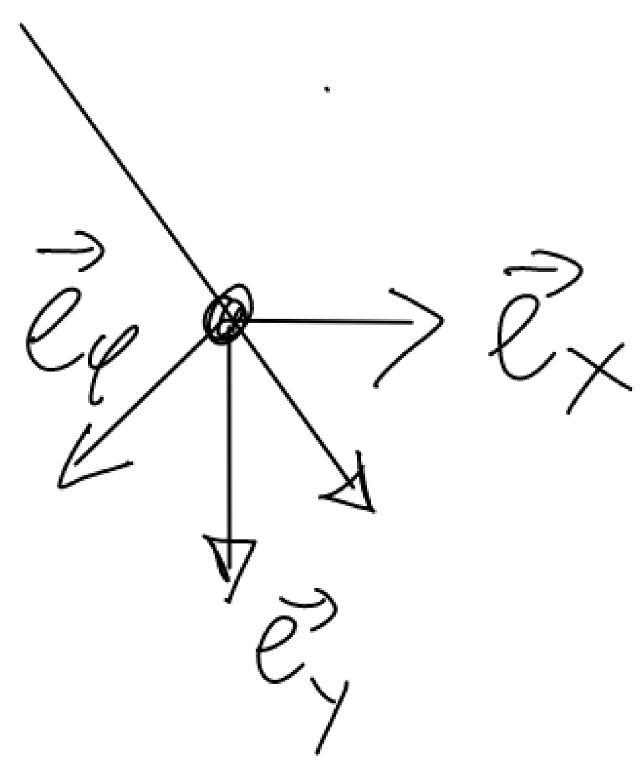
5. Classical dynamics

last lecture new toy (Runge-Kutta method) $\rightsquigarrow$ let's play

we already studied harmonic oscillator, now





$$\vec{e}_r = \sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y$$
$$\vec{e}_\varphi = -\cos\varphi\,\vec{e}_x + \sin\varphi\,\vec{e}_y$$
$$\vec{e}_y = \cos\varphi\,\vec{e}_r + \sin\varphi\,\vec{e}_\varphi$$

$$\vec{x} = l\cdot\vec{e}_r \qquad \dot{\vec{x}} = l\cdot\dot\varphi\cdot\vec{e}_\varphi$$

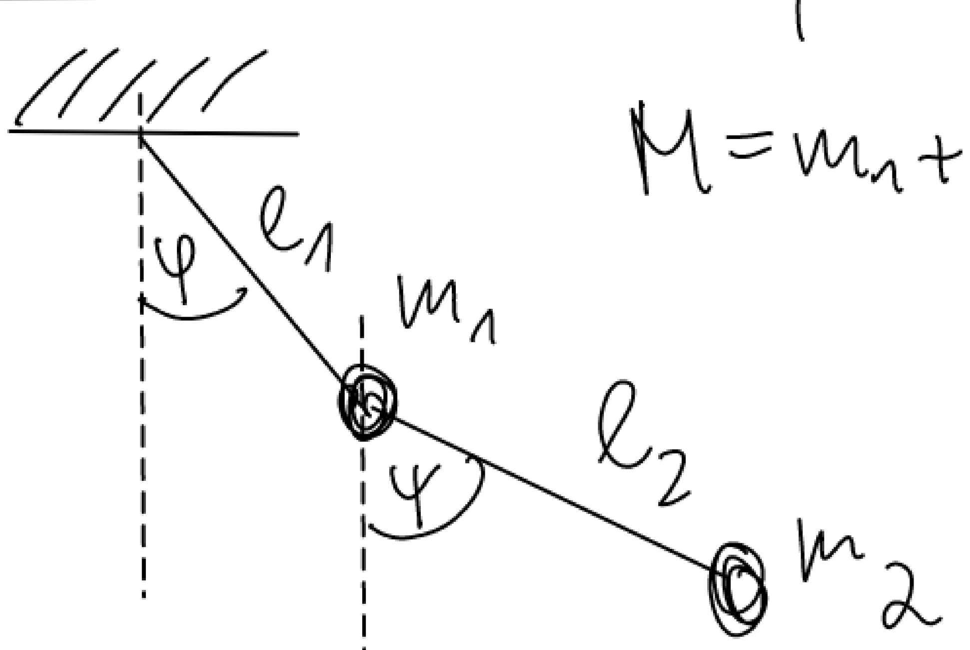
$$\ddot{\vec{x}} = l\cdot\left(\ddot\varphi\,\vec{e}_\varphi + \dot\varphi\,\dot{\vec{e}}_r\right)$$

$\leftarrow$ compensated by rod

$$\rightsquigarrow \quad \ddot\varphi = -\frac{m\cdot g}{l}\sin\varphi = \underbrace{-\frac{m\cdot g}{l}}_{\omega^2}\left(\varphi - \frac{\varphi^3}{3!} + \dots\right)$$

and solve numerically. Nice, but can we do more?

5.1. Double pendulum



$M = m_1 + m_2$ Equations of motion (Eom)?

Classical mechanics $\rightsquigarrow$ Lagrangian

$$L = \frac{1}{2} M l_1^2 \dot\varphi^2 + \frac{1}{2} m_2 l_2^2 \dot\psi^2 + m_2 l_1 l_2 \dot\varphi\dot\psi\cos(\psi-\varphi) - M g l_1(1-\cos\varphi) - m_2 g l_2 (1-\cos\psi)$$

Euler-Lagrange equation give the eom.

$$\boxed{\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot q^i} = 0}$$

$$\ddot\varphi = \left[1-\mu\cos^2(\psi-\varphi)\right]^{-1}\Big[\mu g_1 \sin\psi\cos(\psi-\varphi) + \mu\dot\varphi^2\sin(\psi-\varphi)\cos(\psi-\varphi) - g_1\sin\varphi + \frac{\mu}{\lambda}\dot\psi^2\sin(\psi-\varphi)\Big]$$

and

$$\ddot{\psi} = \left[1 - \mu\cos^2(\psi-\varphi)\right]^{-1}\Big[g_2 \sin\varphi\cos(\psi-\varphi) - \mu\dot{\psi}^2\sin(\psi-\varphi)\cos(\psi-\varphi)$$
$$- g_2\sin\psi - \lambda\dot{\varphi}^2\sin(\psi-\varphi)\Big] \quad \text{with}$$

$\lambda = \ell_1/\ell_2$ $\quad g_i = g/\ell_i$ $\quad$ and $\quad \mu = m_2/M$

Again, let's solve it numerically. Very complicated dynamics. No numerical artifact! Can we understand it better?

5.2. Poincaré-section

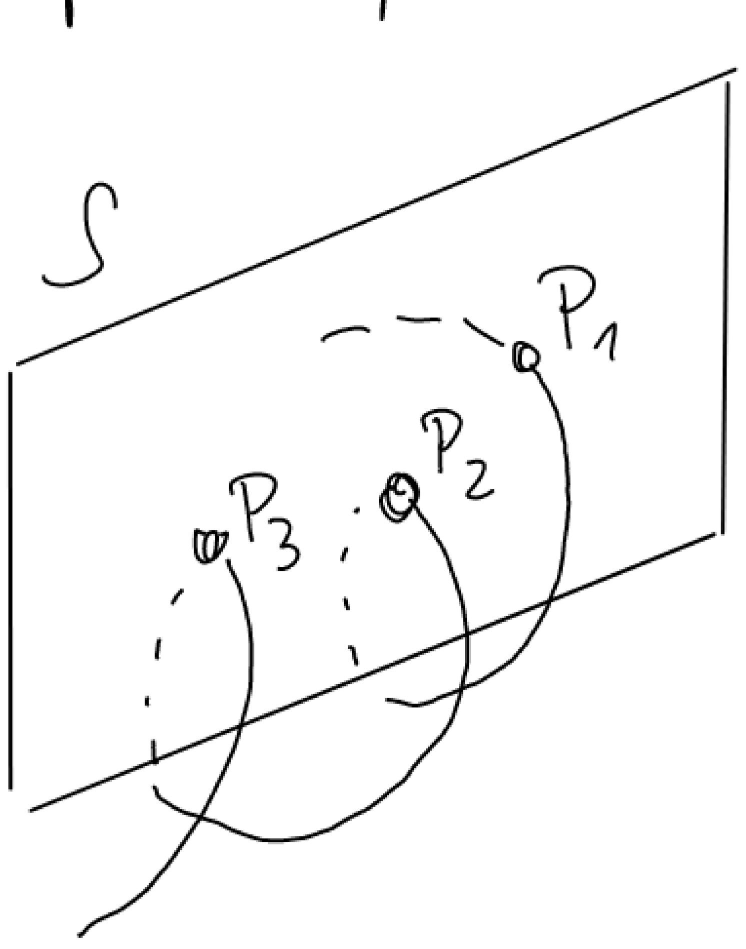
Problem: plot quantities which describe the dynamics

$$\underbrace{\varphi, \psi}_{\text{position}}, \underbrace{\dot{\varphi} \text{ and } \dot{\psi}}_{\text{velocity / momentum}} \quad \text{(position and speed)}$$

position & velocity / momentum = phase space

For 2f degrees of freedom define (2f-1)-dimensional hypersurface S (with orientation) = Poincaré-section and take its intersection with time evolution.

phase-space



⇝ result: a map T, Poincaré-map,

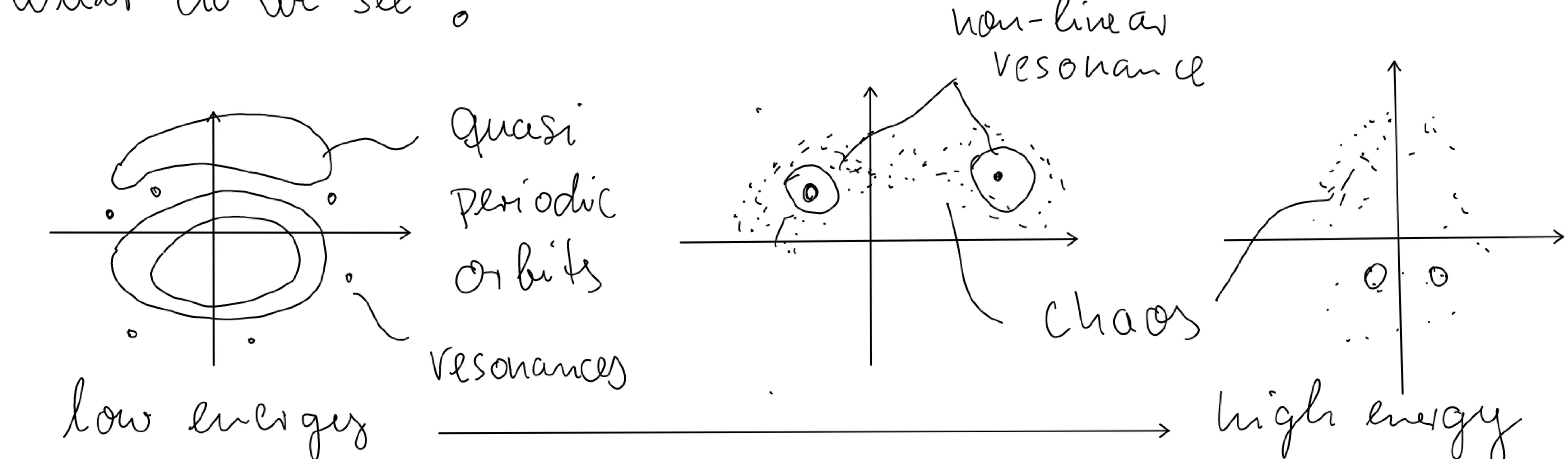
$$P_n \to P_{n+1} = T(P_n)$$

reduces dim. by 1, + energy conservation by 2

For the double pendulum: $\quad \psi = 0$

and $\quad \dot{\psi} + \lambda\dot{\varphi}\cos\varphi > 0$

What do we see?



transition from integrable to chaos

5.3. Integrable dynamics

Def.: A mechanical system with f degrees of freedom ($2f$-dimensional phase space) and f conserved charges in involution is integrable.

conserved charge $F_i(\vec{q},\vec{p})$ with

$$\{F_i, H\} = \sum_{n=1}^{f} \left(\frac{\partial F_i}{\partial q_n} \frac{\partial H}{\partial p_n} - \frac{\partial F_i}{\partial p_n} \frac{\partial H}{\partial q_n} \right) = 0$$

Poisson bracket Hamiltonian position momentum

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \quad \text{and} \quad H = \sum_{n=1}^{f} p_n \dot{q}^n - L$$

in involution: $\{F_i, F_j\} = 0$

$\Rightarrow$ we can introduce action-angle variable

$$I_i(\vec{q},\vec{p}) = I_i(\vec{F}) \qquad \varphi_i(\vec{q},\vec{p})$$

such that

$$\dot{I}_i = -\frac{\partial H}{\partial \varphi_i} = 0 \quad \text{and}$$

$$\dot{\varphi}_i = \frac{\partial H}{\partial I_i} = \omega_i(\vec{I}) = \text{const.}$$

$\Rightarrow \quad \varphi_i(t) = \varphi(0) + \omega_i(\vec{I})\cdot t$

example 2 uncoupled HO's with

$$H = \sum_{i=1}^{2}\left(\frac{p_i^2}{2m_i} + \frac{1}{2}\omega_i^2 q_i^2\right) \qquad F_i = E_i = \frac{p_i^2}{2m_i} + \frac{1}{2}\omega_i^2 q_i^2$$

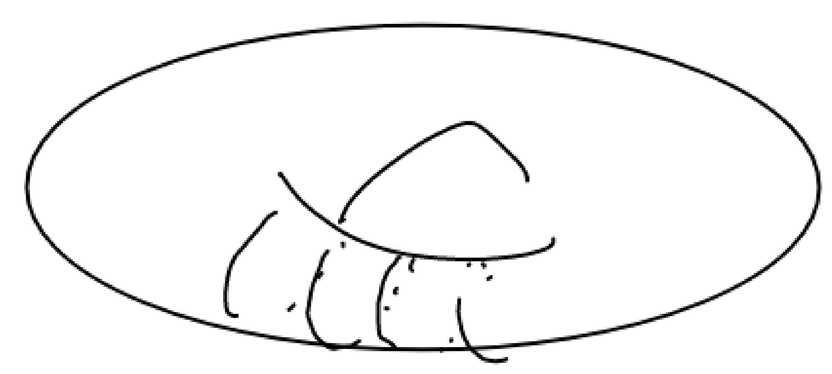
$$I_i = \frac{1}{\omega_i} E_i(q_i, p_i) \quad \text{and} \quad \varphi_i = \arctan\left(\frac{\omega_i q_i}{p_i}\right)$$

if ω_1 and ω_2 are incommensurable, i.e.

$$m_1\omega_1 + m_2\omega_2 = 0 \Leftrightarrow m_1 = m_2 = 0 \quad \text{where}$$

$$m_1, m_2 \in \mathbb{Z}$$

time evolution traces a torus and Poincaré-sections shows



a quasi periodic orbit, otherwise resonance.

KAM (Kolmogorov, Arnold, Moser) - theorem:

- under small, non-integrable, perturbation most of the invariant tori survive with a small deformation
- only close to resonant orbits tori become instable
- their "number" (measure) can be made as small as one wants by tuning the perturbation