5. Canonical quantisation

Remember 2nd lecture: classical string dynamics is governed by

(1.) Wave equation $\eta^{\alpha\beta}\partial_\alpha\partial_\beta X^\mu = 0$ (e.o.m. for X^μ)

(2.) two constraints $(\mathring{X} \pm X')^2 = 0$

from $T_{\alpha\beta} = 0$ (e.o.m. for $h_{\alpha\beta}$)

5.1. Review Klein-Gordon theory

1.) Expand K-G field $\phi(x)$ in terms of solutions to the e.o.m. (Fourier modes)

2.) Lagrangian → Hamiltonian & $\underbrace{\text{(Poisson-brackets)}}_{\text{PBs}}$

3.) derive from canonical, equal time PBs

$$\{\phi, \phi\} = \{\pi, \pi\} = 0 \quad \& \quad \{\phi(\sigma), \pi(\sigma')\} = \delta(\sigma - \sigma')$$

the PBs for the expansion coefficients

4.) Quantisation ⇝ $\boxed{\{\cdot,\cdot\} \to i\hbar[\cdot,\cdot]}$

commutator for operator

↯ order of operator in Hamiltonian is ambiguous
choose i.e. Weyl quantisation $x \cdot p \to \frac{1}{2}(\hat{x}\cdot\hat{p} + \hat{p}\hat{x})$

Result: ∞ decoupled harmonic oscillator

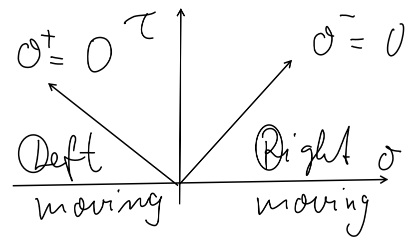
↳ same approach for the string but with constraints

5.2. Mode expansion

Light cone gauge $\sigma^\pm = \tau \pm \sigma$

$$\hookrightarrow \left.\begin{aligned} \partial_\tau &= \partial_+ + \partial_- \\ \partial_\sigma &= \partial_+ - \partial_- \end{aligned}\right\} \text{ e.o.m. } \partial_\tau\partial_\tau X^\mu - \partial_\sigma\partial_\sigma X^\mu = 0$$

$\rightsquigarrow \quad \partial_+ \partial_- X^\mu = 0$

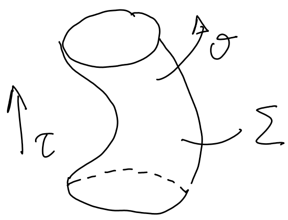


$$X^\mu(\tau,\sigma) = X_R^\mu(\sigma^-) + X_L^\mu(\sigma^+)$$

solve e.o.m., but only defined up to const. shift

$$X_R'(\sigma^-) = 1/2\,(\dot{X} - X')$$
$$X_L'(\sigma^+) = 1/2\,(\dot{X} + X')$$

a) closed strings: $\Sigma = \mathbb{R} \times S^1 \mathrel{\hat=}$ cylinder

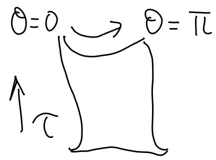


boundary condition $\boxed{X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma + 2\pi)}$

$\rightsquigarrow$
$$X_R'(\sigma^- + 2\pi) = X_R'(\sigma^-)$$
$$X_L'(\sigma^+ + 2\pi) = X_L'(\sigma^+)$$

⚡ Does not apply to the const. shift

b) open string $\Sigma = \mathbb{R} \times [0,\pi] \mathrel{\hat=}$ stripe



$$\boxed{X'(\tau,0) = X'(\tau,\pi) = 0}$$

Dirichlet boundary conditions
$\hat=$ end points of string are fixed

$\longrightarrow$ much more when we look @ D-branes later

for the moment
$$X_L'(\sigma^+) = X_R'(\sigma^- = \sigma^+)$$
$$X_L'(\sigma^+) = X_L'(\sigma^+ + 2\pi)$$

Fourier coefficients:

$$\boxed{\begin{aligned} \alpha_n^\mu &= \sqrt{\tfrac{2}{\alpha'}}\,\frac{1}{2\pi}\int_0^{2\pi} e^{-in\sigma}\, X_R'^\mu(\sigma)\,d\sigma \\ \tilde{\alpha}_n^\mu &= \sqrt{\tfrac{2}{\alpha'}}\,\frac{1}{2\pi}\int_0^{2\pi} e^{in\sigma}\, X_L'^\mu(\sigma)\,d\sigma \end{aligned}}$$

for
a) α_n^μ & $\tilde{\alpha}_n^\mu$ independent
b) $\alpha_n^\mu = \tilde{\alpha}_{-n}^\mu$

complex with $\bar{\alpha}_n = \alpha_{-n}$, $\bar{\tilde{\alpha}}_n = \tilde{\alpha}_{-n}$

5.3. Canonical momentum & Hamiltonian

remember: $\Pi_\mu = \dfrac{\delta \mathcal{L}}{\delta \dot{X}^\mu} = T \cdot \dot{X}^\mu$ with $T = \dfrac{1}{2\pi\alpha'}$

because $\mathcal{L} = -\dfrac{T}{2}\eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu = \dfrac{T}{2}\left(\dot{X}^2 - X'^2\right)$

time evolution $\dfrac{d}{dt} f(X,\pi) = \{H, f(X,\pi)\}$

with Hamiltonian $\boxed{H = \dfrac{1}{2}\int_0^{2\pi} d\sigma' \left(\dfrac{\Pi^2}{T} + T X'^2\right)}$

equal time PBs $\boxed{\{\Pi_\mu(\tau,\sigma), X^\nu(\tau,\sigma')\} = \delta_\mu^\nu\, \delta(\sigma-\sigma')}$

Check: $\{H, X^\mu(\sigma)\} = \int d\sigma' \dfrac{\Pi(\sigma')}{T}\delta(\sigma-\sigma') = \dot{X}^\mu$

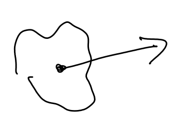
$\{H, \Pi_\mu(\sigma)\} = \int d\sigma'\, T X'_\mu(\sigma')\left(-\delta'(\sigma'-\sigma)\right) = T X''_\mu$

$= \dot{\Pi}_\mu \quad \rightarrow \quad T X''_\mu = T \ddot{X}_\mu$ or

$\ddot{X}_\mu - X''_\mu = 0$ (e.o.m.)

Remember target space Poincaré invariance

⇝ conserved center of mass momentum



P_μ

$P_\mu = \dfrac{1}{2\pi}\int_0^{2\pi} d\sigma\, \Pi_\mu(\sigma)$ conjugate position

$x^\mu = \int_0^{2\pi} d\sigma\, X^\mu(\sigma)$

with $\{P_\mu, x^\nu\} = \delta_\mu^\nu$ (like point particle)

$p^\mu = \dfrac{1}{2\pi}\int_0^{2\pi} d\sigma\, (\partial_+ + \partial_-)\left(X_R^\mu(\sigma^-) + X_L^\mu(\sigma^+)\right)$

$= \dfrac{1}{2\pi}\int_0^{2\pi} d\sigma\, \left(X_L'^\mu(\sigma^+) - X_R'^\mu(\sigma^-)\right)$

$$= \frac{1}{2\pi}\int d\sigma^+ X_R'^{\mu}(\sigma^+) + \frac{1}{2\pi}\int d\sigma^- X_L'^{\mu}(\sigma^-)$$

$$= \sqrt{\frac{\alpha'}{2}}\left(\alpha_0^\mu + \tilde{\alpha}_0^\mu\right)$$

What about α_n^μ & $\tilde{\alpha}_n^\mu$ with $n \neq 0$?

5.4. Harmonic oscillators

Compute: $\{X_R'^{\mu}(\sigma), X_R'^{\nu}(\sigma')\} = \frac{1}{4}\left\{\frac{\Pi^\mu(\sigma)}{T} - X'^{\mu}(\sigma), \frac{\Pi^\nu(\sigma')}{T} - X'^{\nu}(\sigma')\right\}$

$$= \frac{1}{4}\frac{1}{T}\left(\delta'(\sigma-\sigma') + \delta'(\sigma-\sigma')\right)\eta^{\mu\nu} = \pi\alpha'\delta'(\sigma-\sigma')\eta^{\mu\nu}$$

recall $\delta'(-\sigma) = -\delta'(\sigma)$

Same for $\{X_L'^{\mu}(\sigma), X_L'^{\mu}(\sigma')\} = -\pi\alpha'\delta'(\sigma-\sigma')\eta^{\mu\nu}$

and $\{X_L', X_R'\} = 0$

Therefore: $\{\alpha_n^\mu, \alpha_m^\nu\} = \frac{2}{\alpha'}\frac{\eta^{\mu\nu}}{4\pi^2}\pi\alpha'\int_0^{2\pi} d\sigma d\sigma' e^{-in\sigma} e^{-im\sigma'}\delta'(\sigma-\sigma')$

$$= in\,\delta_{n+m}\,\eta^{\mu\nu}$$

Same for $\boxed{\{\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu\} = in\,\delta_{n+m}\,\eta^{\mu\nu}}$

observe: only non-trivial for $m=-n$ frequency

$\rightarrow$ creation & annihilation ops of harmonic oscillator

$$X_R'(\sigma) = \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}} e^{in\sigma}\alpha_n; \quad X_L'(\sigma) = \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}} e^{-in\sigma}\tilde{\alpha}_n$$

Hamiltonian: $H = \ldots = \frac{1}{2}\sum_n\left(\alpha_n\alpha_{-n} + \tilde{\alpha}_n\tilde{\alpha}_{-n}\right)$

$$= \frac{1}{2}\alpha' p^2 + \sum_{n>0}\left(|\alpha_n|^2 + |\tilde{\alpha}_n|^2\right)$$

Last step: integrate $X_{R/L}'(\sigma) \to X_{R/L}(\sigma)$

$$X_L(\sigma^+) = \frac{1}{2}X^\mu + \sqrt{\frac{\alpha'}{2}}\alpha_0^\mu\sigma^+ + \sqrt{\frac{\alpha'}{2}}\, i\sum_{n\neq 0}\frac{\tilde{\alpha}_n^\mu}{n}e^{-in\sigma^+}$$

$$X_R^\mu = \frac{1}{2} x^\mu + \sqrt{\frac{\alpha'}{2}}\, \alpha_0^\mu \sigma^- + \sqrt{\frac{\alpha'}{2}}\, i \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} e^{-in\sigma^-}$$

integration constant

Conclusion: Everything can be written in terms of the zero modes x^μ (with momentum p_μ) and oscillator α_n^μ, $n > 1$. I.e. $J^{\mu\nu} = \dots = 2 p^{[\mu} x^{\nu]}$

Lorentz generator