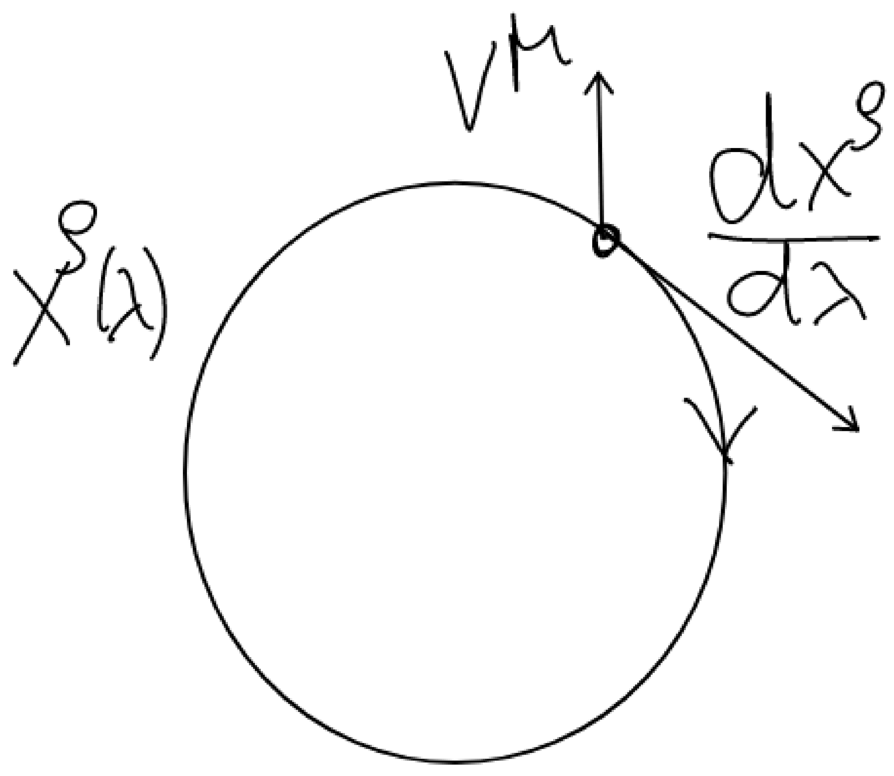
Remember: Matter tells space how to curve and space tells matter how to move. ?

example AdS-space

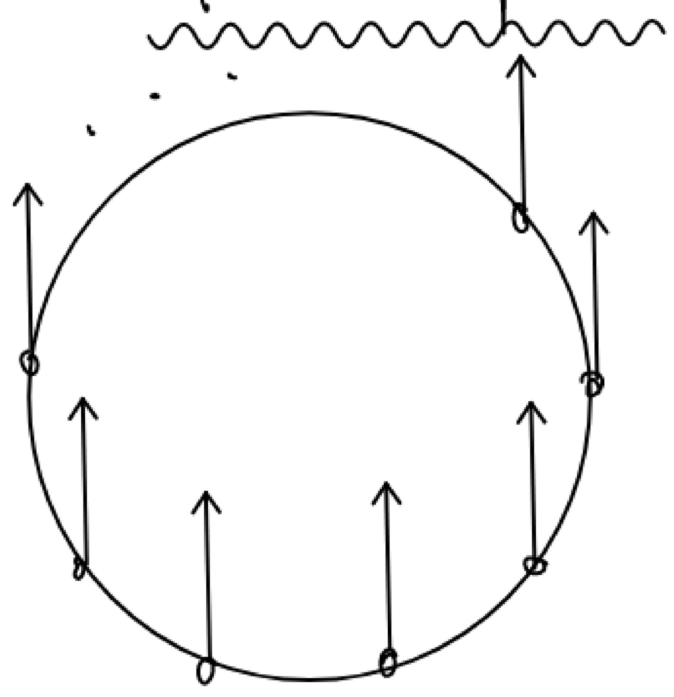
3.1.6 Curvature and parallel transport

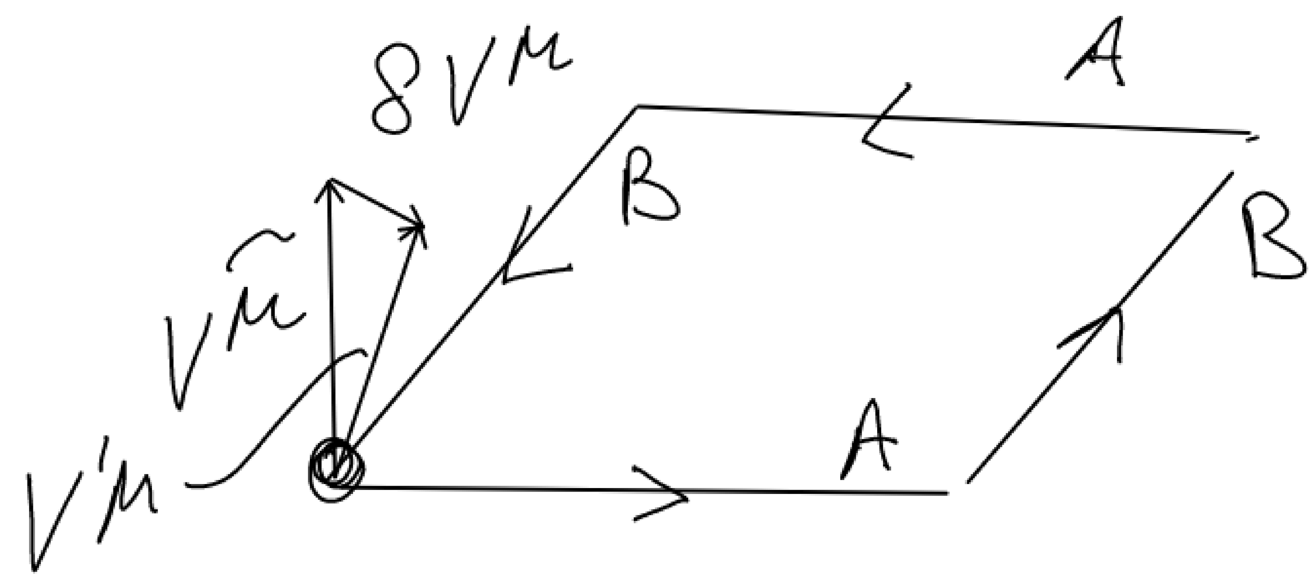


Take a vector V^μ and transport it along a curve according to

$$\frac{dx^\rho}{d\lambda}\nabla_\rho V^\mu = \boxed{\frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}V^\sigma = 0}$$

In flat space, this will keep $V^\mu(\lambda)$'s parallel. Now consider





$\delta V^\mu = V'^\mu - V^\mu$

after and before parallel transport around

infinitesimally $\boxed{\delta V^\rho = R^\rho{}_{\mu\alpha\beta}\, V^\mu A^\alpha B^\beta}$

Riemann Curvature tensor

$[\nabla_\alpha, \nabla_\beta] V^\rho = R^\rho{}_{\mu\alpha\beta} V^\mu - T^\sigma{}_{\alpha\beta} \nabla_\sigma V^\rho$

Torsion tensor $T^\sigma_{\mu\nu} = -2\Gamma^\sigma_{[\mu\nu]}$

$= 0$ for us

$\boxed{R^\rho{}_{\mu\alpha\beta} = 2\partial_{[\alpha}\Gamma^\rho_{\beta]\mu} + 2\Gamma^\rho_{[\alpha|\sigma}\Gamma^\sigma_{|\beta]\mu}}$ measures the curvature of space-time

Similar to the field strength $F_{\alpha\beta}$ in YM-theory also has a Biachi identity $\nabla_{[\lambda} R_{\mu\nu]\alpha\beta} = 0$

symmetries: $R_{\mu\nu\alpha\beta} = R_{[\mu\nu]\alpha\beta} = R_{\mu\nu[\alpha\beta]}$ and

$R_{\mu[\alpha\beta\gamma]} = 0$

$$\begin{array}{c|c|c|c} 3 & d & d+1 & 2 \\ \hline 2 & d-1 & d & 1 \end{array}$$

or $\dfrac{d^2(d+1)(d-1)}{12} = \dfrac{d^2(d^2-1)}{12}$ components

particularly important are $\underbrace{R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}}_{\text{Ricci tensor}}$ $\quad \underbrace{R = g^{\mu\nu}R_{\mu\nu}}_{\text{Ricci/curvature scalar}}$

3.1.7. Geodesic

= shortest connection between two points, consider

arc length $S[x^\mu] = \int ds \sim \sqrt{-g_{\mu\nu}\dfrac{dx^\mu}{d\lambda}\dfrac{dx^\nu}{d\lambda}}$

$$\delta S = 0 \quad \Rightarrow \quad \boxed{\frac{d^2x^\beta}{d\lambda^2} + \Gamma^{\beta}_{\alpha\nu}\frac{dx^\alpha}{d\lambda}\frac{dx^\nu}{d\lambda}} \text{ geodesic equation}$$

Point particle action in curved space (space tells matter how to move)

3.1.8. Einstein-Hilbert action

Remember scalar ED (lecture 1) with

$$\mathcal{L}_G = \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi D^\mu\bar\phi + m^2\phi\bar\phi$$

pure gauge part for gravity

coupling to em from $A_\mu j^\mu$ with $\partial_\mu j^\mu = 0$

$$\mathcal{L} = \underbrace{\frac{\sqrt{-g}}{2\varkappa^2}(R-\Lambda)}_{\text{Einstein-Hilbert action}} + \underbrace{\sqrt{-g}\Big(\nabla_\mu\phi\nabla^\mu\bar\phi + m^2\phi\bar\phi\Big)}_{\text{coupling to gravity}}$$

$\nabla_\mu = \partial_\mu$ for scalar

Field equations are $\underbrace{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}}_{G_{\mu\nu} = \text{Einstein tensor}} + \underset{\text{cosmo const.}}{\Lambda} g_{\mu\nu} = \kappa^2 T_{\mu\nu}$

Energy-Momentum tensor

$$\underbrace{\nabla^\mu G_{\mu\nu} = 0}_{\text{because of BIs}} = \nabla^\mu T_{\mu\nu}$$

3.2. Maximal symmetric spaces

Killing vector fields: $\mathcal{L}_\xi g_{\mu\nu} = 0$ capture symmetries of space time

$$\mathcal{L}_{\xi_1} \mathcal{L}_{\xi_2} g_{\mu\nu} - \mathcal{L}_{\xi_2} \mathcal{L}_{\xi_1} g_{\mu\nu} = \mathcal{L}_{\xi_3} g_{\mu\nu} = 0$$

$\xi_3 = [\xi_1, \xi_2] = \mathcal{L}_{\xi_1} \xi_2 \rightsquigarrow$ isometry Lie algebra/group

i.e. Minkowski space d transtions + $\frac{d(d-1)}{2}$ rotations/boosts

$\hat{=}$ $d(d+1)/2$ isometries (max in d dimensions)

$\rightsquigarrow$ curvature is the same everywhere in every direction

$$R_{\mu\nu\rho\sigma} = \frac{\overset{\sim \text{const.}}{R}}{d(d-1)} \left(g_{\nu\sigma} g_{\mu\rho} - g_{\nu\rho} g_{\mu\sigma} \right)$$

three options:

$R > 0$	de Sitter space time
$R < 0$	anti-de Sitter -"-
$R = 0$	Minkowski -"-

3.3. Anti-de Sitter space time

similar to S^2 example, by embedding into $\mathbb{R}^{d,2}$

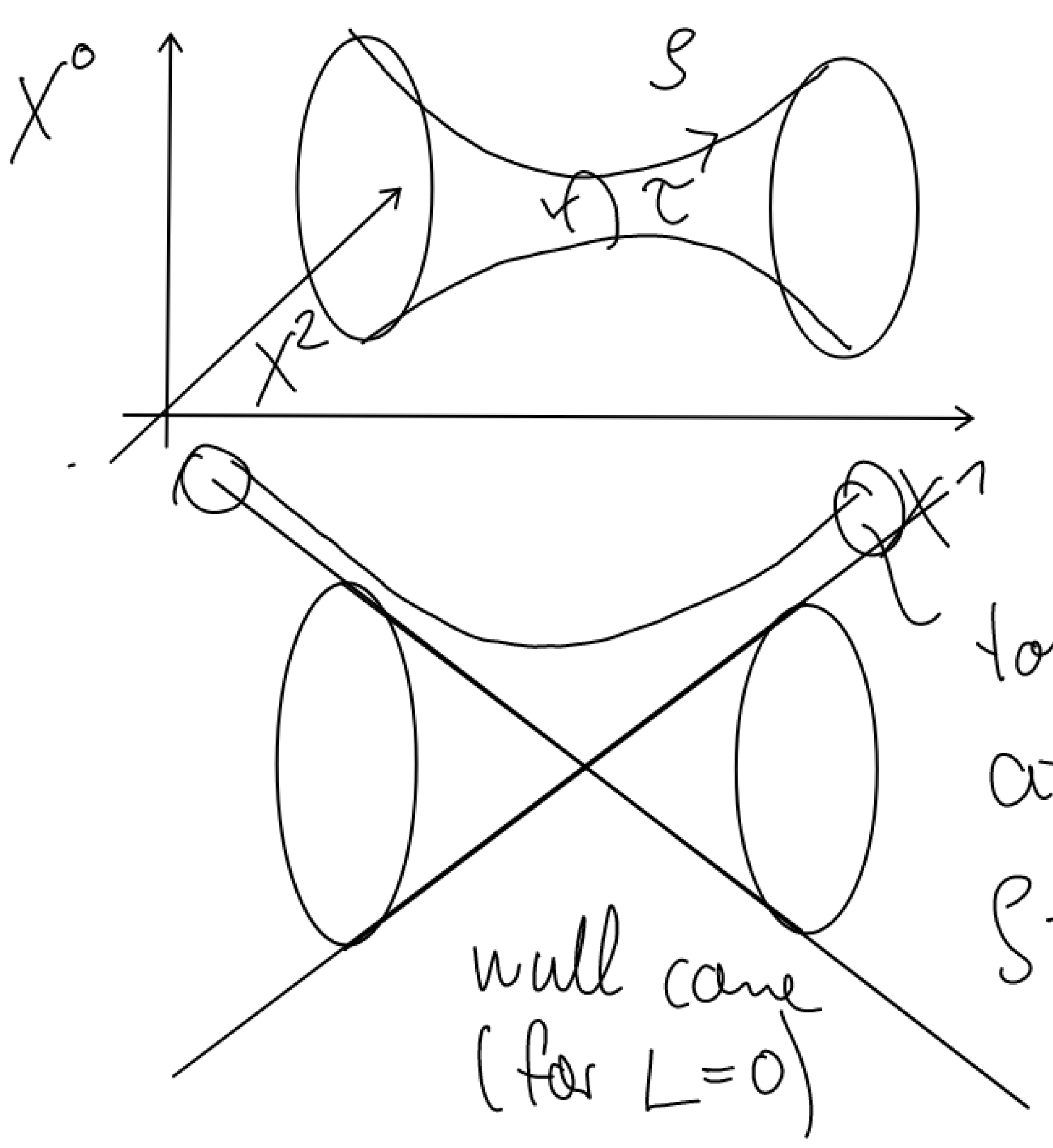
$(X^0, \ldots, X^{d+2})$ with $\tilde{\eta}_{MN} = \mathrm{diag}(-, +, \ldots, +, -)$

as hypersurface $\tilde{\eta}_{MN} X^M X^N = \boxed{-(X^0)^2 + \sum_{i=1}^{d} (X^i)^2 - (X^{d+1})^2 = -L^2}$

$\rightsquigarrow$ $O(d,2)$ rotations leave it invariant $\rightsquigarrow$ isometry group

<u>global coordinates</u>

$$X^0 = L\cosh\rho\cos\tau$$
$$X^{d+1} = L\cosh\rho\sin\tau$$
$$X^i = L\,\Omega_i\sinh\rho \quad \text{for } i=1,\dots,d$$
$$\sum_i \Omega_i^2 = 1 \sim S^{d-1}$$



touches at $\rho\to\pm\infty$ $\Big\}$ conformal boundary = all null lines through the origin

$$\partial AdS_{d+1} = \{\, [X] \mid X\in\mathbb{R}^{d,2},\ X\neq 0,\ \tilde\eta_{MN}X^M X^N = 0 \,\}$$

$[X]$ identifies $X^N \sim \lambda X^N$

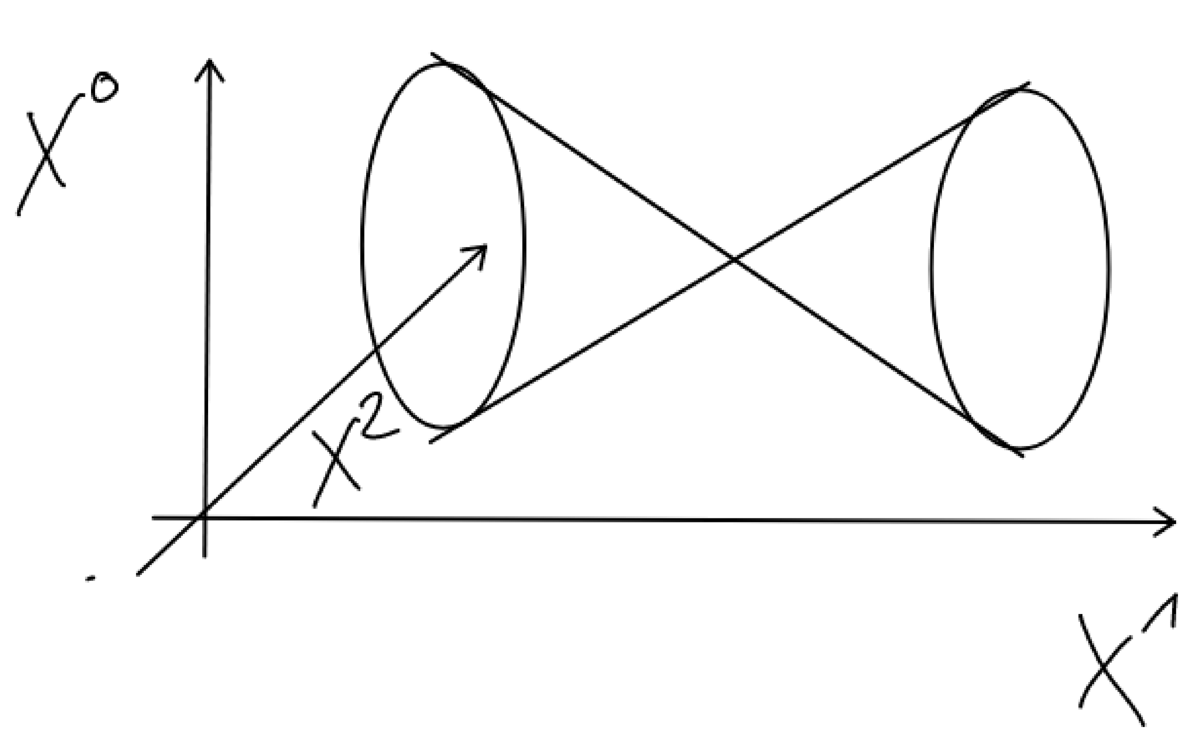
Points of ∂AdS_{d+1} have to satisfy

$$\underbrace{\sum_{i=1}^{d}(X^i)^2 = 1}_{S^{d-1}} \quad\text{and}\quad \underbrace{(X^0)^2+(X^{d+1})^2 = 1}_{S^1}$$

but $[X] = [-X]$ although different on $S^{d-1}\times S^1$

$$\rightsquigarrow \boxed{\partial AdS_{d+1} = \left(S^{d-1}\times S^1\right)/\mathbb{Z}_2 = d\text{-dim compactified Minkowski space}}$$

<u>Why?</u>



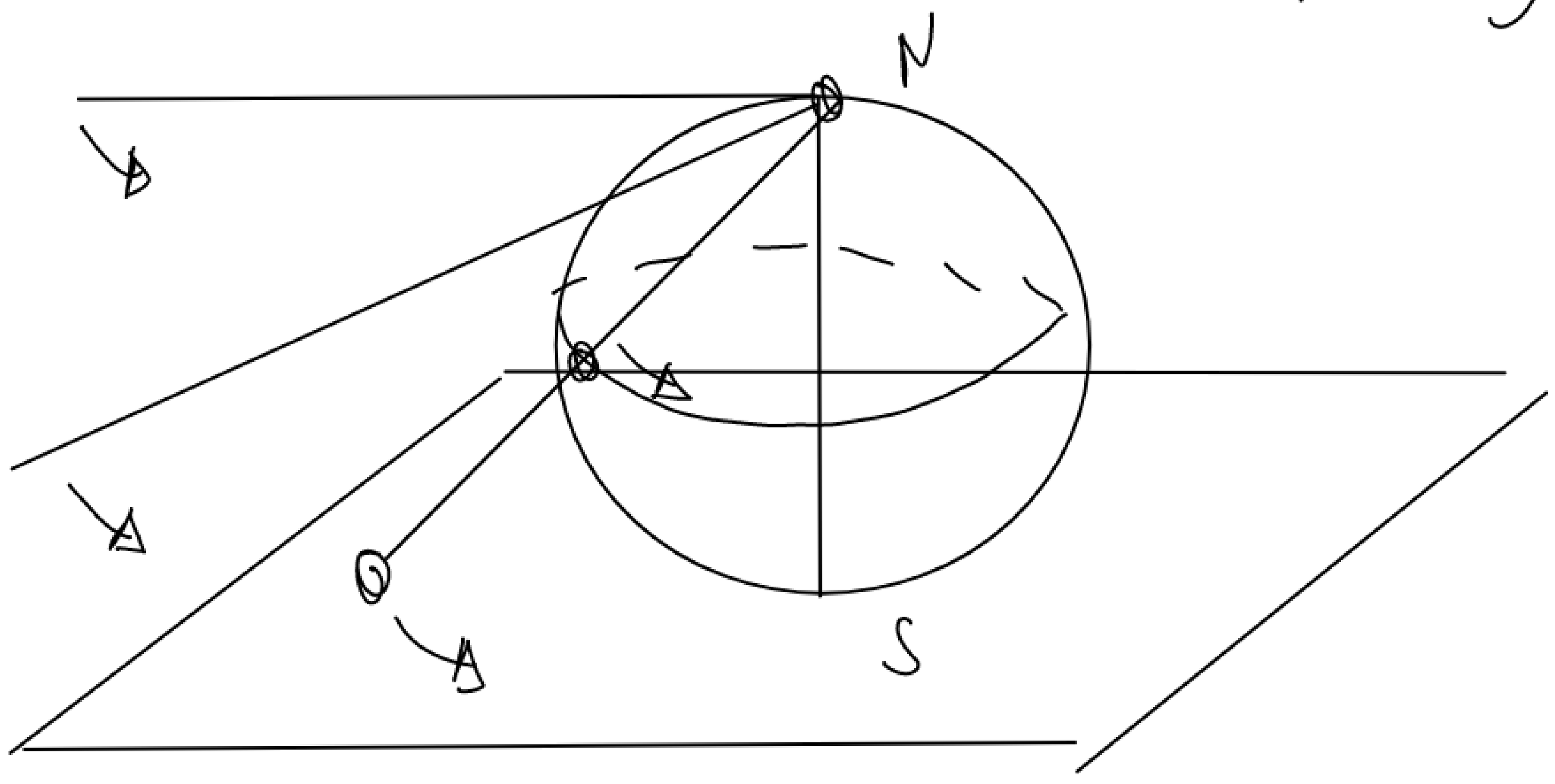
points on the null cone can be written as $uv = \underbrace{\eta_{\mu\nu}}_{\text{diag}(-1,+1,\dots,+1)} X^\mu X^\nu \sim \nu = (0,\dots,d-1)$

<u>for $v\neq 0$</u> we fix λ in $[X]$ such that $v = +1$ or -1

next we solve for u. $\mapsto$ all values for X^μ are allowed
= Minkowski space

for $v=0$ additional light cone

compare with conformal mapping of S^2 to $\mathbb{R}^2$



$S \to \infty$
$N \to (0,0)$

$S^2 = \underbrace{\mathbb{R}^2 \cup \{\infty\}}_{\text{compactification of } \mathbb{R}^2}$