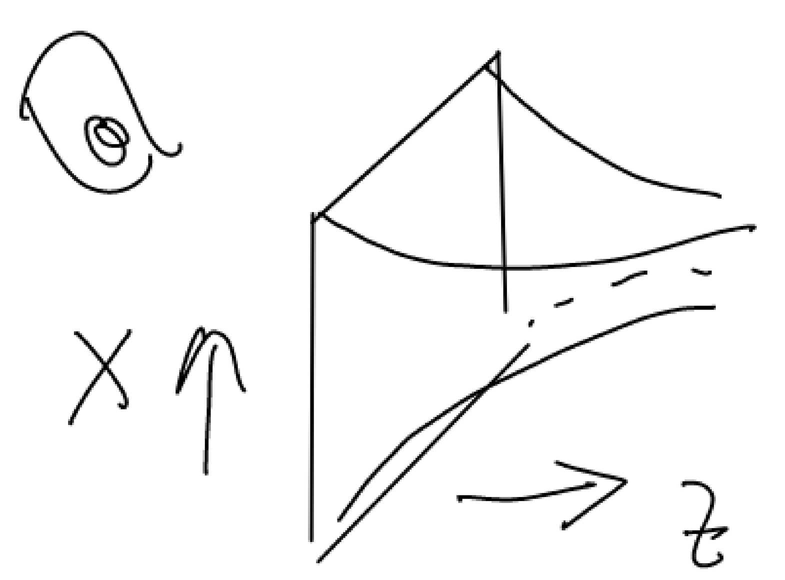
11. Holographic renormalization



z in AdS seems to be related to an engery (cut-off) scale.
$\rightsquigarrow$ renormalization ($\nearrow$ 4.4.)

scalar from last lecture in Fefferman - Graham coord.
with $ds^2 = L^2\left(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\delta_{\mu\nu}\,dx^\mu dx^\nu\right)$ boundary @ $\rho = 0$

ansatz for eom solution: $\phi(x,\rho) = \rho^{(d-\Delta)/2}\,\bar\phi(x,\rho)$
with $\bar\phi(x,\rho) = \phi_{(0)}(x) + \rho\,\phi_{(2)}(x) + \rho^2\phi_{(4)}(x) + \dots$

$$\rightsquigarrow \boxed{\phi_{(2n)} = \frac{1}{2n(2\Delta - d - 2n)}\,\Box_0\,\phi_{(2n-2)}} \quad \delta^{\mu\nu}\partial_\mu\partial_\nu$$

$\lightning$: Problem for $2\Delta - d - 2k = 0$ for any $k \in \mathbb{N}$
i.e. $k = 1$ with $\Delta = \frac{1}{2}(d-2)$ $\rightsquigarrow$ modify ansatz

- $\bar\phi(x,\rho) = \phi_{(0)} + \rho\left(\phi_{(2)} + \ln\rho\,\chi_{(2)}\right)$ with

$\chi_{(2)} = -1/4\,\Box_0\,\phi_{(0)}$,

$\phi_{(2)}$ more complicated, non-locally (∞ ∂'s) related to $\phi_{(0)}$

for general $k \in \mathbb{N}$ $\quad \chi_{(2k)} = -\frac{1}{2^{2k}\,\Gamma(k)\,\Gamma(k+1)}\left(\Box_0\right)^k \phi_{(0)}$

like before into the action, only boundary contributes

$$S_{reg} = -\frac{C}{2}\int d^dx\,\sqrt{g}\,g^{\rho\rho}\,\phi\,\partial_\rho\phi\,\Big|_{\rho=\varepsilon}$$

$$= C L^{d-1} \int d^d x \Big(\underbrace{\varepsilon^{-\Delta+\frac{d}{2}} a_{(0)}}_{= -\frac{1}{2}(d-\Delta)\,\phi_{(0)}^2} + \underbrace{\varepsilon^{-\Delta+\frac{d}{2}+1} a_{(1)}}_{= -(d-\Delta+1)\,\phi_{(0)}\phi_{(2)}} + \ldots - \ln\varepsilon \; a_{(2\Delta-d)} \Big) \quad \ldots$$

for $\Delta > d/2$ S_{reg} diverges for $\varepsilon \to 0$

Σ c compensate by adding counter terms to the action.

$$S_{ct} = \frac{C}{L} \int d^d x \underbrace{\sqrt{\gamma}}_{\gamma_{\mu\nu} = L^2 \delta_{\mu\nu}/\varepsilon} \left(\frac{d-\Delta}{2} \phi^2(x,\varepsilon) + \frac{1}{2(2\Delta-d-2)} \phi(x,\varepsilon) \Box_\gamma \phi(x,\varepsilon) \right)$$

here for $d/2 < \Delta < d/2 + 1$, for larger Δ more ct

now the renormalized action $S_{ren} = \lim\limits_{\varepsilon \to 0} S_\varepsilon$

with $S_\varepsilon = S_{reg} + S_{ct}$

plugging in the solution for $\phi(x)$ we get the finite contributions

$$S_{reg} = -\frac{d\, C L^{d-1}}{2} \int d^d x \; \phi_{(0)}(x)\, \phi_{(+)}(x) + \ldots$$

$$S_{ct} = C L^{d-1} (d-\Delta) \int d^d x \; \phi_{(0)}(x)\, \phi_{(+)}(x) + \ldots$$

both together give

$$S_\varepsilon = \frac{C L^{d-1}}{2} (d - 2\Delta) \frac{\Gamma(\Delta)}{\pi^{d/2}\, \Gamma(\Delta - \frac{d}{2})} \int d^d x \int d^d y \; \frac{\phi_{(0)}(x)\, \phi_{(0)}(y)}{(x-y)^{2\Delta}}$$

and we again get

$$\langle \mathcal{O}(x)\, \mathcal{O}(y) \rangle = C L^{d-1} (2\Delta - d) \frac{\Gamma(\Delta)}{\pi^{d/2}\, \Gamma(\Delta - d/2)} \frac{1}{(x-y)^{2\Delta}}$$

Remark: Same method of constructing counter terms also applies to the metric.