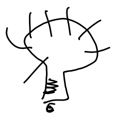
11. BRST Symmetry

Becchi, Rouet, Stora and Tyutin

remember: Quantisation of (non-)abelian gauge field required gauge fixing $\rightsquigarrow$ we loose gauge symmetry
Today, we restore it with BRST.

11.1. The Faddeev-Popov Lagrangian

goal: evaluate path integral $I = \int \mathcal{D}A \exp\left[-\frac{i}{4}\int d^4x\, (F^a_{\mu\nu})^2\right]$



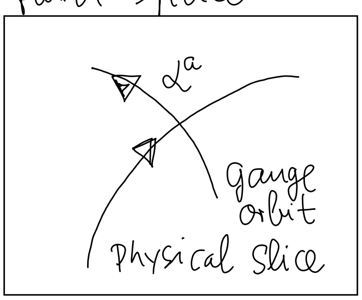
trick remove redundancy by gauge fixing cond.

$G(A) = 0$ and

$$1 = \int \mathcal{D}\alpha(x)\, \delta(G(A^\alpha))\, \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)$$

with $(A^\alpha)^a_\mu t_a = e^{i\alpha^a t_a}\left[A^b_\mu t_b + \frac{i}{g}\partial_\mu\right] e^{-i\alpha^c t_c}$

field space



and infinitesimal version

$$(A^\alpha)^a_\mu = A^a_\mu + \frac{1}{g}\partial_\mu\alpha^a + f_{bc}{}^a A^b_\mu \alpha^c$$
$$= A^a_\mu + \frac{1}{g}D_\mu\alpha^a \qquad (1)$$

covariant derivative acting on α^a in adjoint representation

now we can write:

$$I = \left(\int\mathcal{D}\alpha\right)\int\mathcal{D}A\, e^{i\underbrace{S[A]}_{=-\frac{1}{4}\int d^4x\,(F^a_{\mu\nu})^2}}\,\delta(G(A))\,\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)$$

in generalised Lorentz gauge, $G(A) = \partial^\mu A^a_\mu(x) - \omega^a(x)$, (2)

we eventually find:

$$\langle A^a_\mu(x) A^b_\nu(y)\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{-i}{k^2+i\varepsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\delta^{ab} e^{-ik(x-y)}$$

remember: $\xi = 1$ is called Feynman-'t Hooft gauge

⚡ Do not forget the det(...)! Because we have

$$\frac{\delta G(A^\alpha)}{\delta \alpha} \overset{(1),(2)}{=} \frac{1}{g} \partial^\mu D_\mu \,, \quad \text{instead of just a constant factor (in abelian version)}$$

⟶ results in ghost field Lagrangian

$$\boxed{\det\left(\frac{1}{g}\partial^\mu D_\mu\right) = \int \mathcal{D}c\,\mathcal{D}\bar{c}\, \exp\left[i\int d^4x \underbrace{\bar{c}\,(-\partial^\mu D_\mu)\,c}_{}\right]}$$

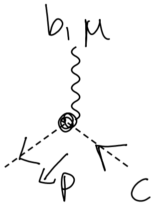
ghost $\hat{=}$ violates spin-statistic, i.e. $\mathcal{L}_{ghost}$
Spin 0 but fermionic path integral ⟹ c = Grassmann var.

$$\mathcal{L}_{ghost} = \bar{c}^a\left(-\partial^2 \delta_a^c - g f_{ab}{}^c \partial_\mu A^{b\mu}\right) c_c$$

with propagator: $\langle c_a(x)\, \bar{c}^b(y)\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2} \delta_a^b\, e^{-ik(x-y)}$

and Feynman rules:

$$a \dashleftarrow b = \frac{i\delta_a^b}{p^2} \,,$$



$$= -g f_{ab}{}^c p_\mu$$

11.2. The BRST Lagrangian

Question: Can we still find the orginal gauge transformations in the gauge fixed action?

Idea: Introduce a new (commuting) scalar field B_a

$$\mathcal{L} = -\tfrac{1}{4}\left(F_{\mu\nu}^a\right)^2 + \bar{\psi}(i\not{D} - m)\psi - \frac{\xi}{2}(B_a)^2 + B_a \partial^\mu A_\mu^a + \bar{c}^a\left(-\partial^\mu (D_\mu)_a{}^b\right) c_b$$

B_a has no kinetic term ⟶ auxiliary field
we can integrate it out

This Lagrangian has the global symmetry:

$$\boxed{\begin{aligned} \delta A^a_\mu &= \varepsilon (D_\mu)^{ab} c_b \\ \delta \psi &= i g \varepsilon c_a t^a \psi \end{aligned}}$$

$$\delta c_a = -\frac{1}{2} g \varepsilon f^{bc}{}_a c_b c_c$$

$$\delta \bar{c}^a = \varepsilon B^a$$

$$\delta B^a = 0$$

local gauge transformation with $\alpha^a(x) = g \varepsilon c^a(x)$

comm. anti-comm.

We denote this transformation with

$$\delta \phi := \varepsilon Q \phi \quad \text{any field, i.e.} \quad Q A^a_\mu = (D_\mu)^{ab} c_b$$

One can now check:

$$\boxed{Q^2 \phi = 0}, \text{ i.e. } \quad Q^2 c_a = \frac{1}{2} g^2 f^{be}{}_a f^{cd}{}_e c_b c_c c_d$$

Jacobi identity $\longrightarrow = 0$

In the Hamiltonian picture Q commutes with the Hamilton operator H, $\quad [Q, H] = 0$, $\quad Q^2 = 0$

$\longrightarrow$ eigenstates of H decompose into:

1) $|\psi_1\rangle$ with $Q|\psi_1\rangle \neq 0$
2) $|\psi_2\rangle$ with $|\psi_2\rangle = Q|\psi_1\rangle$ for a particular $|\psi_1\rangle$
3) $|\psi_0\rangle$ with $Q|\psi_0\rangle = 0$ and $|\psi_0\rangle \neq Q|\psi_1\rangle$ for any $|\psi_1\rangle$

and the Hilbert space decomposes into $\mathcal{H}_i \in |\psi_i\rangle$

For single particle states: $\mathcal{H}_1$ has forward gauge bosons and anti-ghosts

$\mathcal{H}_2$ has backward gauge bosons and ghosts

physical fields $\longrightarrow$ $\boxed{\mathcal{H}_0 \text{ has transverse gauge bosons}}$

compare with exterior derivative: $d\phi = \partial_\mu \phi \, dx^\mu$ with

$d^2 = 0$ and $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{\deg \omega_2} \omega_1 \wedge d\omega_2$

$$\left.\begin{array}{lll} d\omega = 0 & \hat{=} & \text{closed } n\text{-form} \\ \omega = d\lambda & \hat{=} & \text{exact } n\text{-form} \end{array}\right\} n = \deg \omega$$

de Rham cohomology: $H^k_{dR}(M) = \dfrac{\text{closed}}{\text{exact}}$ k-forms

manifold

analogy: $d \hat{=} Q$

$$\boxed{\text{physical states are defined by BRST cohomology}}$$