10.2.4. Correlation functions

remember last lecture 2 bc for bulk gravity

$\phi_{(+)}$ $\langle \phi \rangle$ of CFT operator

$\phi_{(0)}$ source for this — " — ↗ 4.2.

$\boxed{\text{CFT}}$ $S'_{CFT} = S_{CFT} - \int d^dx \, \phi_{(0)}(x) \, \mathcal{O}(x)$

$\boxed{\text{AdS}}$

$$\boxed{\begin{aligned} Z[\phi_{(0)}] &= e^{-W[\phi_{(0)}]} = \left\langle e^{\int d^dx \, \phi_{(0)}(x) \, \mathcal{O}(x)} \right\rangle_{CFT} \\ W[\phi_{(0)}] &= S_{SUGRA}[\phi] \Big|_{\lim_{z \to 0} \phi(z,x) z^{\Delta - d} = \phi_{(0)}} \end{aligned}}$$

weak form of the correspondence

remember W generates connected correlators

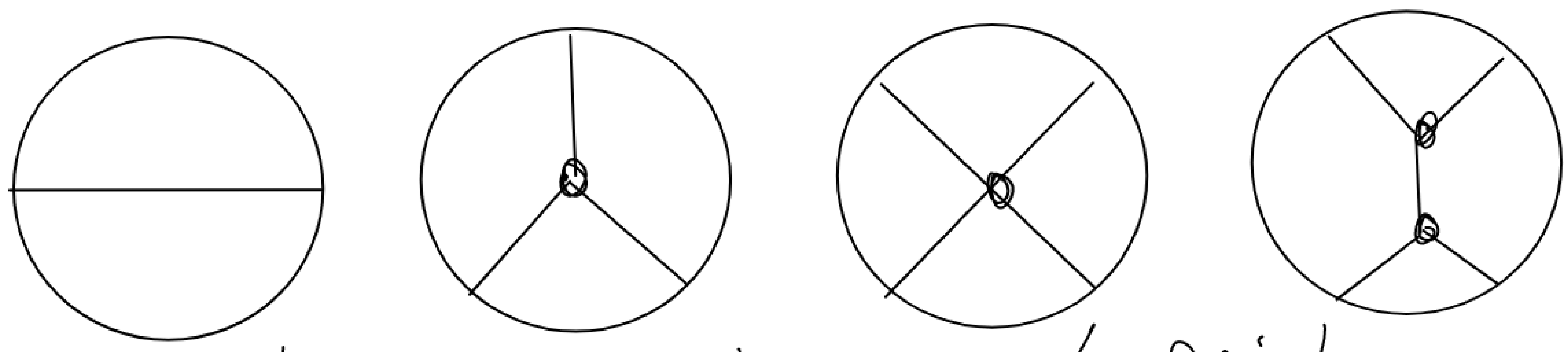
$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle_{CFT,c} = - \frac{\delta^n W}{\delta \phi^1_{(0)}(x_1) \dots \delta \phi^n_{(0)}(x_n)} \Bigg|_{\phi^i_{(0)} = 0}$$

recipe
1) find bulk field ϕ for operator $\mathcal{O}$ of scaling dimension Δ
2) solve SUGRA eom. for ϕ under bc
   $\phi(x,z) = z^{d-\Delta} \phi_{(0)}(x)$ for $z \to 0$
3) insert solution into $S_{SUGRA} = W$
4) Variation wrt. $\phi_{(0)}$

strong form $\langle \dots \rangle_{CFT} = Z_{string} \Big| \dots$ not know explicitly

10.3. Witten diagrams (no loops)

recipe $\hat{=}$ compute tree-level diagrams in AdS space pictorial



2-point 3-point 4-point

} Witten diagrams

→ we need 1) bulk-bulk propagator

2) bulk-boundary —"—

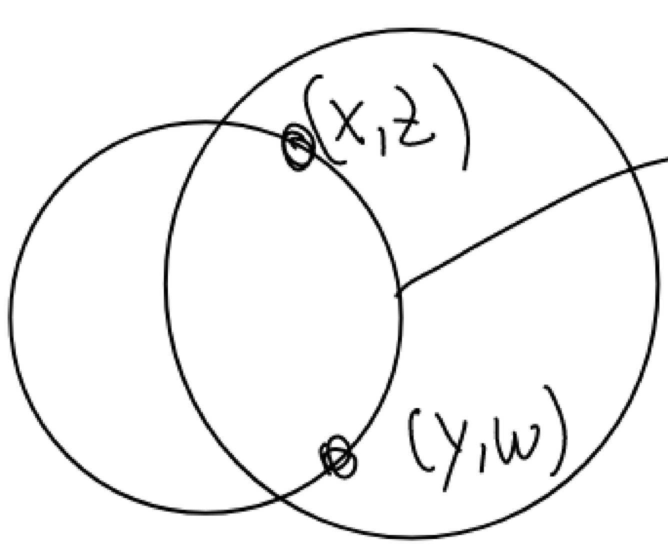
1) AdS Klein-Gordon equation

$$\left(\Box_g - m^2\right) \phi_\Delta(x,z) = \underbrace{J(x,z)}_{\text{sources}}$$

g: AdS-metric, Euclidean for regularity

$$\left(\Box_g - m^2\right) G_\Delta(x,z;y,w) = \frac{\delta(z-w)\,\delta^d(x-y)}{\sqrt{g}}$$

G_Δ: complicated



d = chordal distance = length of geodesic

$$= \int_{(x,z)}^{(y,w)} ds = \ln\left(\frac{1+\sqrt{1-\xi^2}}{\xi}\right)$$

$$\xi = \frac{2zw}{z^2+w^2+(x-y)^2}$$

$$C_\Delta = \frac{\Gamma(\Delta)}{\pi^{d/2}\,\Gamma(\Delta - d/2)}$$

hyper geo. function

$$G_\Delta(\xi) = \frac{C_\Delta}{2^\Delta(2\Delta-d)}\,\xi^\Delta \cdot {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta+1}{2}; \Delta - \frac{d}{2}+1; \xi^2\right)$$

2) $$K_\Delta(x,z;y) = \lim_{w\to 0} \frac{2\Delta - d}{w^\Delta}\, G_\Delta(x,z;y,w)$$

$$= C_\Delta \left(\frac{z}{z^2+(x-y)^2}\right)^\Delta \quad \text{because then}$$

@ the boundary $\lim_{z\to 0}\left(z^{\Delta-d} K_\Delta(x,z;y)\right) = \delta^d(x-y)$

$\lightning$ divergences close to the boundary, like in

two-point function

$$S[\phi] = \frac{C}{2} \int dz\, d^dx \sqrt{g} \left(g^{mn} \partial_m \phi \partial_n \phi + \underbrace{m^2}_{= L^{-2}\Delta(\Delta - d)} \phi^2 \right)$$

if ϕ solves eom $S[\phi]$ only the

boundary term $S[\phi] = -\frac{C}{2} \int d^dx \sqrt{g}\, g^{zz} \phi \partial_z \phi \Big|_{z=\varepsilon}$

($z \to \infty$ contribution vanishes, $\varepsilon \to 0$)

BUT $\sqrt{g}\, g^{zz} = \left(\frac{L}{z}\right)^{d-1}$ diverges for $z \to 0$

regularize! ($\nearrow$ 4.4) Take a finite ε!

1) Fourier transformation $\phi(x,z) = \int \frac{d^dp}{(2\pi)^d} e^{ipx} \phi(p,z)$

2) exact solution to ($\nearrow$ last lecture)

$$z^2 \partial_z^2 \phi(p,z) - (d-1) z \partial_z \phi(p,z) - (m^2L^2 + p^2 z^2) \phi(p,z) = 0,$$

not just asymptotics

Bessel functions

$$\rightsquigarrow \phi(p,z) = A_p z^{d/2} K_\nu(z|p|) + B_p z^{d/2} I_\nu(z|p|)$$

$\nu = \Delta - d/2$

$B_p = 0$ because $I_\nu \to \infty$ for $z \to \infty$

A_p fixed by asymptotics to get

$$\phi(p,z) = \frac{z^{d/2} K_\nu(z|p|)}{\varepsilon^{d/2} K_\nu(z|p|)} \phi_{(0)}(p)\, \varepsilon^{d-\Delta}$$

in the action this gives

$$S[\phi] = -\frac{CL^{d-1}}{2\varepsilon^{d-1}} \int \frac{d^dp}{(2\pi)^d} \frac{d^dq}{(2\pi)^d} (2\pi)^d \delta^d(p+q) \phi(p,z) \partial_z \phi(q,z)$$

with this we can finally compute

$$\langle \mathcal{O}(p)\mathcal{O}(q)\rangle = -(2\pi)^{2d}\frac{\delta^2 S[\phi_{(0)}]}{\delta\phi_{(0)}(-p)\,\delta\phi_{(0)}(-q)}$$

$$= -\frac{(2\pi)^d\delta^d(p+q)\,CL^{d-1}}{\varepsilon^{2\Delta-d}}\left(\frac{d}{2}+\frac{\varepsilon|p|\,K_\nu'(\varepsilon|p|)}{K_\nu(\varepsilon|p|)}\right)$$

for positive $\nu=\Delta-d/2$, by expanding K_ν, we get

$$\langle \mathcal{O}(p)\mathcal{O}(q)\rangle = -(2\pi)^d\delta^d(p+q)\,CL^{d-1}\frac{(-1)^{\nu+1}}{2^{2(\nu-1)}\Gamma(\nu)^2}|p|^{2\nu}\ln(\varepsilon|p|)$$

Now, just take the $|p|^{2\nu}\ln(\varepsilon|p|)$ part and transform to position space!

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = CL^{d-1}\frac{\Gamma(\Delta)}{\Gamma(\Delta-d/2)}\frac{2\Delta-d}{\pi^{d/2}|x-y|^{2\Delta}}$$

comes from the SUGRA expansion ($\uparrow$ 10.2.2)

matches two point correlation function in 6.3.

Contrived $\rightsquigarrow$ better holographic renormalization!