9. Gravity and String Theory

Today more about my research.

9.1 General Relativity

Idea: Construct a theory which is invariant under coordinate changes.

$$x^\mu \to x^\mu + \xi^\mu \qquad \phi(x^\mu) \to \phi(x^\mu) + \xi^\nu \partial_\nu \phi(x^\mu)$$

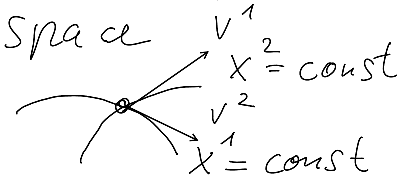
infinitesimal & coordinate dependent $\to$ local symmetry

vector $\quad V^\mu \to V^\mu + L_\xi V^\mu$

$$L_\xi V^\mu = \xi^\nu \partial_\nu V^\mu - V^\nu \partial_\nu \xi^\mu$$

transformation of the tangent space

Lie derivative



Scalar $\quad \phi \to \phi + L_\xi \phi$

$$L_\xi \phi = \xi^\mu \partial_\mu \phi$$

Leibnitz rule: $\quad L_\xi(\underbrace{V^\mu w_\mu}_{\text{scalar}}) = (L_\xi \underbrace{V^\mu}_{\text{vector}}) w_\mu + V^\mu (L_\xi \underbrace{w_\mu}_{\text{one-form}})$

one-form $\quad w_\mu \to w_\mu + L_\xi w_\mu$

$$L_\xi w_\mu = \xi^\nu \partial_\nu w_\mu + w_\nu \partial_\mu \xi^\nu$$

While $\quad \delta_\xi \{\phi, V^\mu, w_\mu\} = L_\xi \{\phi, V^\mu, w_\mu\}$

$$\delta_\xi (\partial_\mu V^\nu) = \partial_\mu (\delta_\xi V^\nu) \neq L_\xi (\partial_\mu V^\nu)$$

We know this problem from gauge theories.

$\to$ Solution: covariant derivative

$$\boxed{\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\rho}{}^\nu V^\rho}$$

connection

calculate field strength with

$$[\nabla_\mu, \nabla_\nu] V^\rho := R^\rho{}_{\sigma\mu\nu} V^\sigma - T_{\mu\nu}{}^\lambda \nabla_\lambda V^\rho$$

two contributions:

torsion $T_{\mu\nu}{}^{\lambda} = 2\,\Gamma^{\lambda}_{[\mu\nu]}$ and

<u>curvature</u> $R^{\rho}{}_{\sigma\mu\nu} = 2\left(\partial_{[\mu}\Gamma^{\rho}_{\nu]\sigma} + \Gamma^{\rho}_{[\mu|\lambda|}\Gamma^{\lambda}_{\nu]\sigma}\right)$

similar to the fieldstrength $F_{\mu\nu}^{i}\,t_i$ in Yang-Mills

↗ section 4.2.

Question: How do we compute $\Gamma^{\rho}_{\mu\nu}$?

Answer: I) set torsion $T_{\mu\nu}{}^{\lambda} = 0$

II) require $\nabla_{\rho}\, g_{\mu\nu} = 0$ (metric compatible)

$$= \partial_\rho g_{\mu\nu} - \Gamma^{\lambda}_{\rho\mu}\, g_{\lambda\nu} - \Gamma^{\lambda}_{\rho\nu}\, g_{\mu\lambda}$$

$$\Rightarrow \boxed{\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}\, g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)}$$

Levi-Civita connection

Two more important quantities:

Ricci tensor: $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$ and

Ricci scalar: $R = R_{\mu\nu}\, g^{\mu\nu} = R_{\mu}{}^{\mu}$

used to construct invariant

$$\boxed{S_{EH} = \frac{1}{2\kappa}\int d^4x\, \sqrt{-\det(g_{\mu\nu})}\, R}$$ Einstein-Hilbert action

9.2. Quantisation

Superficial degree of divergence $D = (d-2)\underbrace{L}_{\text{number of loops}} + 2(1-\underbrace{N}_{\text{external legs}})$

↯ non-renormalisable for $d > 2$

⇝ look at $d = 2$

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\, \sqrt{-\overset{\det g_{\mu\nu}}{g}}\,\left(R + \partial_\mu X^i\, \partial^\mu X^j\, G_{ij}\right)$$

Einstein-Hilbert action in 2d does not depend on metric!

$\rightsquigarrow$ topological

To have propagating, local degrees of freedom add scalar fields X^i $i = 1, \dots, n$ with couplings $G_{ij}(X)$

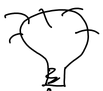
After gauge fixing of $g_{\mu\nu}$:

$$\boxed{S = \frac{1}{4\pi\alpha'} \int d^2\sigma \; G_{ij}(X)\, \partial_\mu X^i \partial^\mu X^j} \quad \text{with } g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

non-linear σ-model

9.3. One-loop β-functions

We need perturbative expansion!



Pick any point $\bar{X}^i$ and expand $G_{ij}(X)$ around it.

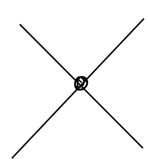
$$G_{ij}(X) = \delta_{ij} - \frac{\alpha'}{3} R_{ikjl}(\bar{X})\, Y^k Y^l + O(Y)$$

$$X^i = \bar{X}^i + \sqrt{\alpha'}\, Y^i$$

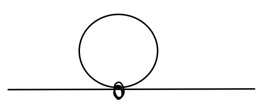
$\hat{=}$ adopt Riemann normal coordinates

$$\Rightarrow S = \frac{1}{4\pi} \int d^2\sigma \left[\partial_\mu Y^i \partial^\mu Y^j \delta_{ij} - \frac{\alpha'}{3} R_{ikjl}\, Y^k Y^l \partial_\mu Y^i \partial^\mu Y^j \right]$$

Feynman rules: ——— $\sim \dfrac{\delta^{ij}}{k^2}$

 $\sim R_{ikjl}\, k^i k^j$

one-loop correction to propagator:

 $\sim R_{ikjl} \displaystyle\int \frac{d^d k}{(2\pi)^d} \frac{\delta^{ij}}{k^2} \overset{\text{dim. reg.}}{\sim} \frac{R_{ij}}{\varepsilon} + \text{finite}$

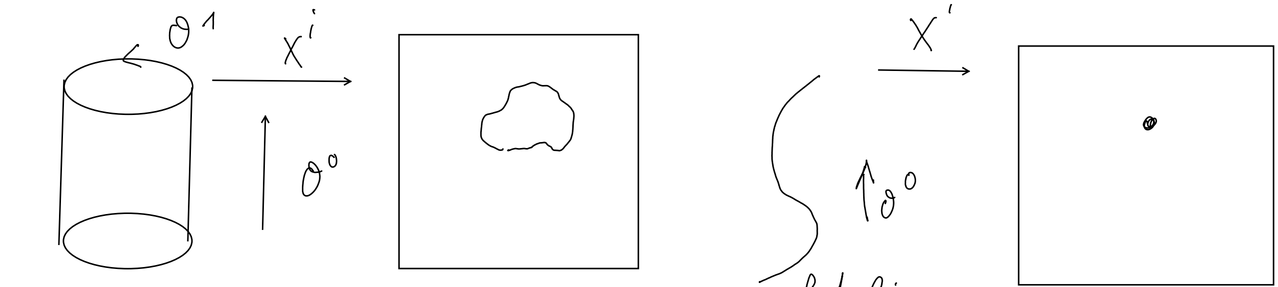
apply our tools for one-loop β-function

$$\boxed{\beta_{ij}(G) = \alpha' R_{ij}}$$

conformal fix point at $R_{ij} = 0 \Leftrightarrow \delta S_{EH} = 0$

higher loop corrections to β-function
give higher derivative corrections to EH action.

Interpretation of σ-model



world sheet target space world line

vs point particle

String $\leftrightarrow$ quantum gravity