Surprisingly Complex Punctures from a Dynamical System

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in collaboration with

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at CHAPEL HILL

Theories of class S [Gaiotto, 2012]

6D $\mathcal{N} = (2,0)$ SCFT

- IIB on $\mathbb{R}^{5,1} \times \mathbb{C}^2/\Gamma$, $\Gamma \subset SU(2) \rightarrow$ ADE-classification [Witten, 1995]
- N M5-branes in flat space (A_N) [Strominger, 1996]

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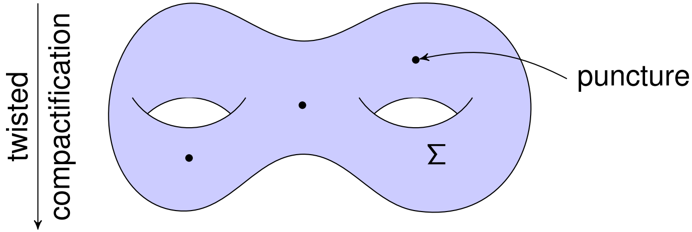

4D $\mathcal{N} = 2$ SCFT

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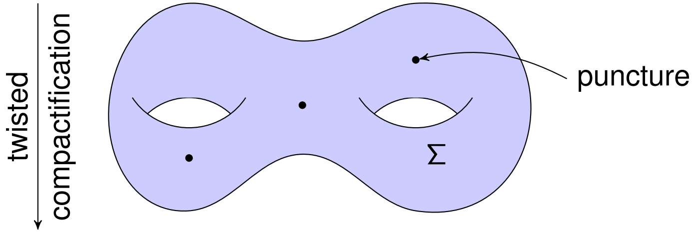
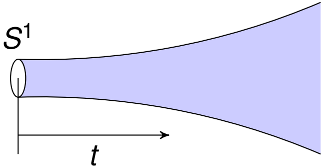


4D $\mathcal{N} = 2$ SCFT

- gauge group G
- flavor symmetry from punctures on Σ

Constrains on punctures

- compactify 6D $\mathcal{N} = (2,0)$ on S^1



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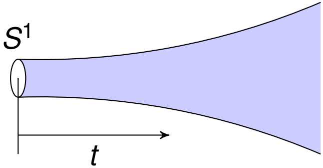

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- maximally SUSY puncture $\rightarrow$ 1/2 BPS equations for

$$\Sigma(t) = \frac{\Sigma}{t} \qquad Q(t) = \frac{Q}{t} \qquad \tilde{Q}(t) = 0$$

(in terms of $\mathcal{N}$=1 4D superfields)

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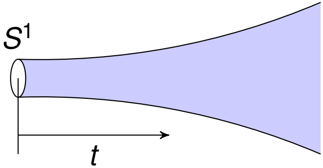

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- Σ, Q, $Q^\dagger$ are representations of $\mathfrak{su}(2)$

¿ Generalization to $\mathcal{N}$=1 ?

- 6D $\mathcal{N} = (1, 0)$ SCFT
- compactification Σ with punctures $\rightarrow$ 4D $\mathcal{N}$=1 SCFTs [Razamat, Vafa, and Zafrir, 2016]

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Challenges

- much more 6D $\mathcal{N} = (1, 0)$ than $\mathcal{N} = (2, 0)$ SCFTs [Heckman, Morrison, and Vafa, 2014]
- less constrained by SUSY

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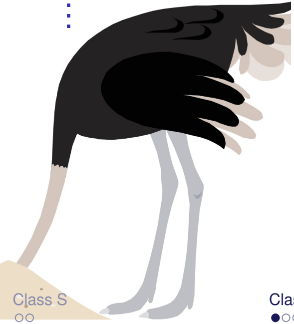

- use "simple" 6D $\mathcal{N}$=(1,0) SCFT

  N M5-branes probing ADE-singularity $\mathbb{C}^2/\Gamma$
- try to classify all punctures [Heckman, Jefferson, Rudelius, and Vafa, 2016]
- harder than you might think

Theories of Class S_Γ... [Heckman, Jefferson, Rudelius, and Vafa, 2016]

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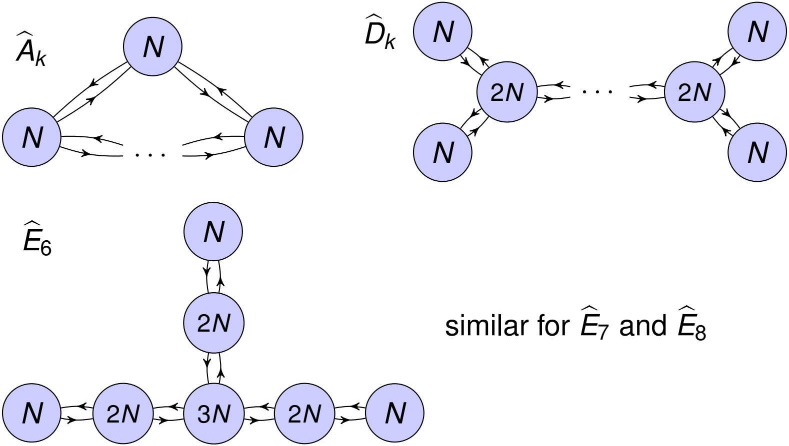
- organized according to extended Dynkin diagrams



similar for $\widehat{E}_7$ and $\widehat{E}_8$

... and their punctures

- again, maximally SUSY punctures $\rightarrow$ 1/2 BPS equations for

$$\Sigma(t) = \frac{\Sigma}{t} \qquad Q(t) = \frac{Q}{t} \qquad \tilde{Q}(t) = \frac{\tilde{Q}}{t}$$

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(in terms of $\mathcal{N}$=1 4D superfields in covering space)

- results in generalized Nahm pole equations [Heckman, Jefferson, Rudelius, and Vafa, 2016]

$$[\Sigma, Q] = Q \qquad [Q, \tilde{Q}] = 0$$

$$[\Sigma, \tilde{Q}] = \tilde{Q} \qquad [Q, Q^\dagger] + [\tilde{Q}, \tilde{Q}^\dagger] = \Sigma$$

plus invariance under Γ-action with

doublet $\begin{pmatrix} Q \\ \tilde{Q} \end{pmatrix}$ and singlet Σ

A closer look at $\widehat{A}_k$ quivers

- choose $\Gamma \ni \gamma = \mathrm{diag}(1_N, \omega 1_N, \omega^2 1_N, \ldots, \omega^k 1_N)$

$$\gamma Q \gamma^\dagger = \omega Q \qquad \gamma \tilde{Q} \gamma^\dagger = \omega^{-1} \tilde{Q} \qquad \gamma \Sigma \gamma^\dagger = \Sigma$$

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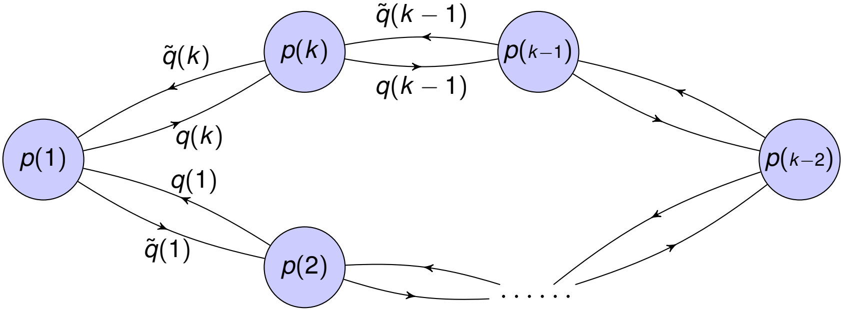
$$\Sigma = \begin{pmatrix} p(1) & & \\ & \ddots & \\ & & p(k) \end{pmatrix} \quad Q = \begin{pmatrix} & q(1) & & \\ & & \ddots & \\ & & & q(k-1) \\ q(k) & & & \end{pmatrix} \quad \tilde{Q} = \begin{pmatrix} & & & \tilde{q}(k) \\ \tilde{q}(1) & & & \\ & \ddots & & \\ & & \tilde{q}(k-1) & \end{pmatrix}$$



$\mathcal{N}$=1 $\widehat{A}_k$ quivers and a dynamical system

- rewrite gen. Nahm pole eq. in terms of $q(i)$, $\tilde{q}(i)$ and $p(i)$

$$
\begin{aligned}
[Q, \tilde{Q}] = 0 \quad &\to \quad q(i+1)\tilde{q}(i+1) = q(i)\tilde{q}(i) \\
[Q, Q^\dagger] + [\tilde{Q}, \tilde{Q}^\dagger] = \Sigma \quad &\to \quad x(i) - x(i-1) = p(i) \\
[\Sigma, Q] = Q \quad &\to \quad q(i)\Big(p(i) - p(i+1)\Big) = q(i) \\
[\Sigma, \tilde{Q}] = \tilde{Q} \quad &\to \quad -\tilde{q}(i)\Big(p(i) - p(i+1)\Big) = \tilde{q}(i)
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with $x(i) = q(i)q(i)^* - \tilde{q}(i)\tilde{q}(i)^*$

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$$[Q,\tilde{Q}]=0 \quad\rightarrow\quad q(i+1)\tilde{q}(i+1)=q(i)\tilde{q}(i)$$

$$[Q,Q^\dagger]+[\tilde{Q},\tilde{Q}^\dagger]=\Sigma \quad\rightarrow\quad x(i)-x(i-1)=p(i)$$

$$[\Sigma,Q]=Q \quad\rightarrow\quad q(i)\Big(p(i)-p(i+1)\Big)=q(i)$$

$$[\Sigma,\tilde{Q}]=\tilde{Q} \quad\rightarrow\quad -\tilde{q}(i)\Big(p(i)-p(i+1)\Big)=\tilde{q}(i)$$

with $x(i)=q(i)q(i)^*-\tilde{q}(i)\tilde{q}(i)^*$

- Q is nilpotent, thus Q^k=$1_k \prod_{i=1}^{k} q(i)$=0 $\quad\rightarrow\quad q(i)\tilde{q}(i)=0$
- knowing $x(i)$ is sufficient to get $q(i)$ and $\tilde{q}(i)$

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- discrete dynamical system

$$f : \begin{pmatrix} p \\ x \end{pmatrix}(i+1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} p \\ x \end{pmatrix}(i) - \operatorname{sgn} x(i)$$

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- choose $x(1)$, $p(1)$ and all other $x(i)$, $p(i)$ are fixed[1]

1 In general $p(i+1)$ is unconstrained if $x(i) = 0$. We choose $p(i+1)$=$p(i)$ to formally extend the dynamical system beyond this point.

Periodic orbits

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- strongly depends on the initial condition, e.g.

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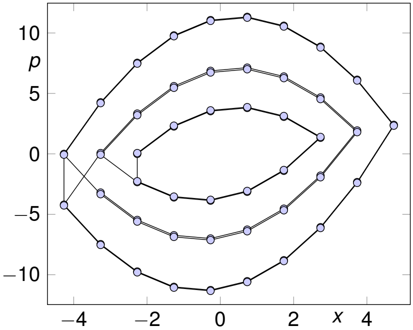

$k = 100,\ x(1) = -\frac{48}{15},\ p(1) = -\frac{49}{15}$

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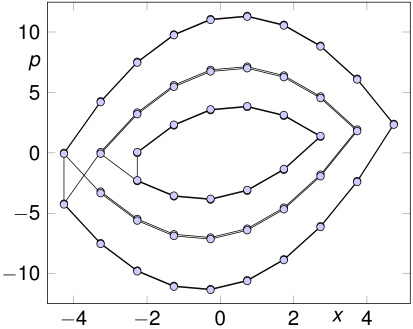
$x(1) = -\frac{13}{4}, \; p(1) = -\frac{12}{4}$



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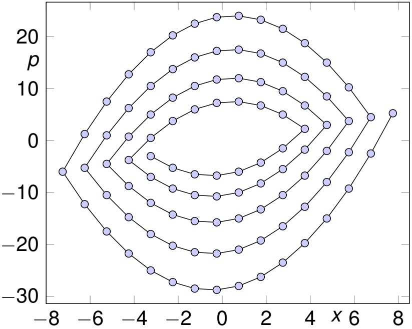

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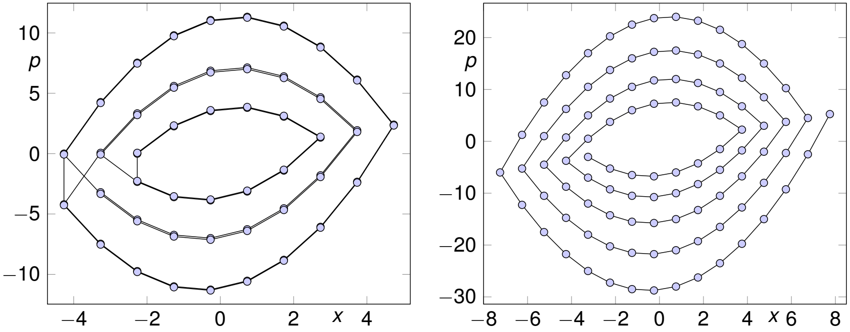

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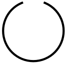
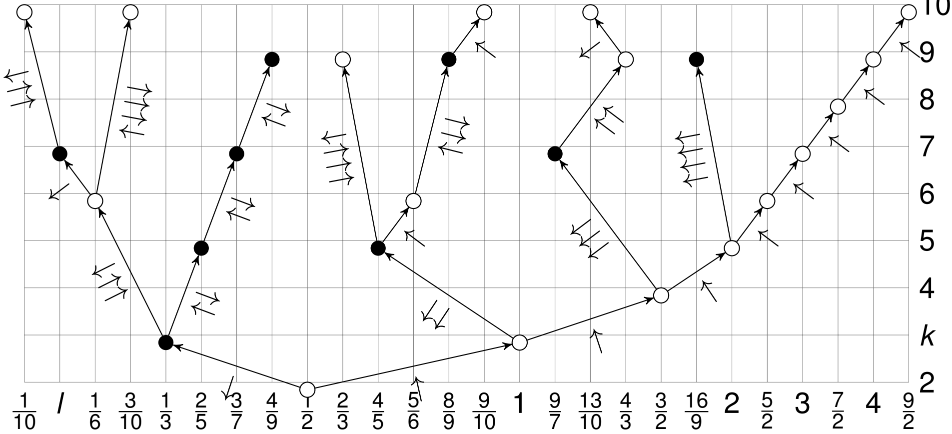
- How to find the right initial conditions?

A tree of solutions

- periodic orbits of type  $x(k) = 0$ organized in tree structure



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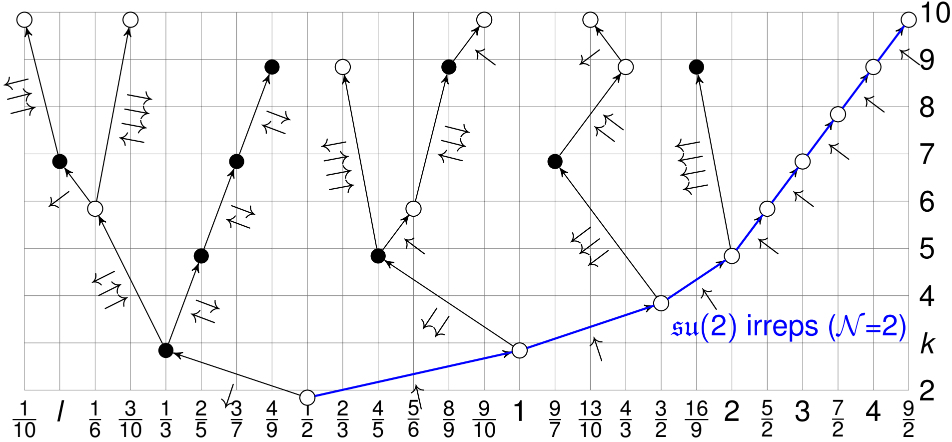

and qualitative

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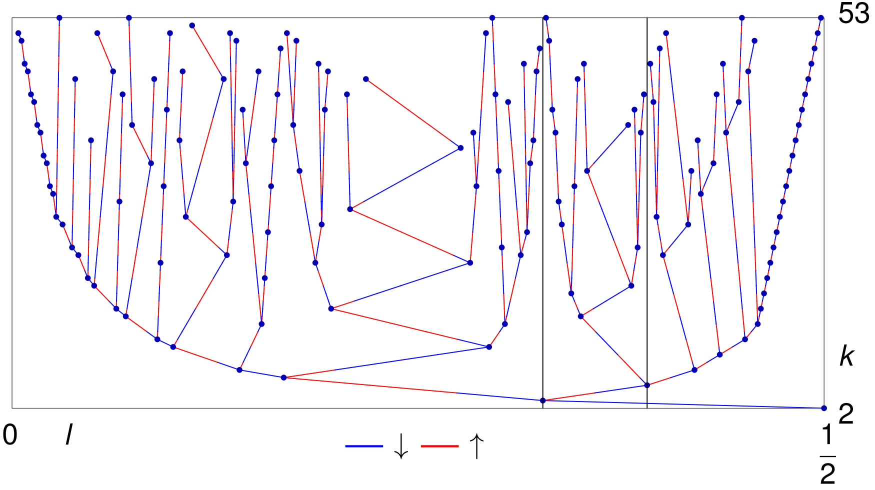

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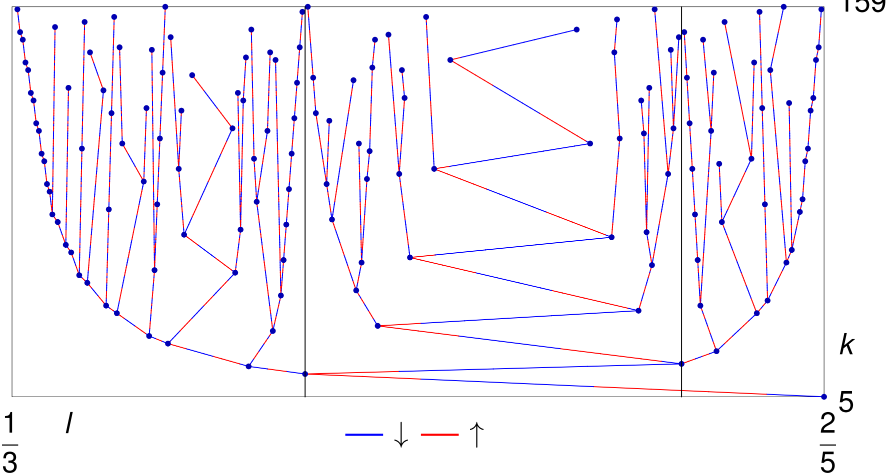

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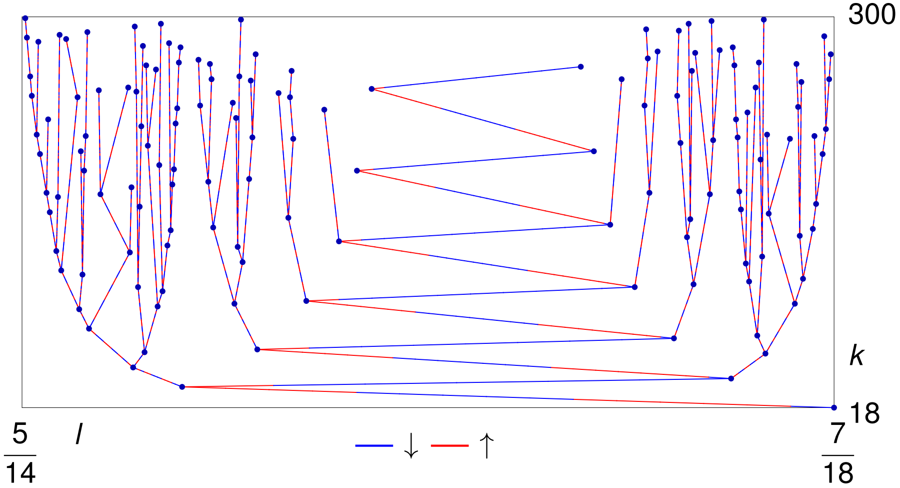

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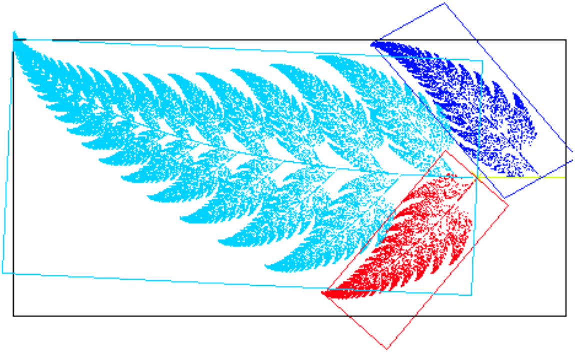

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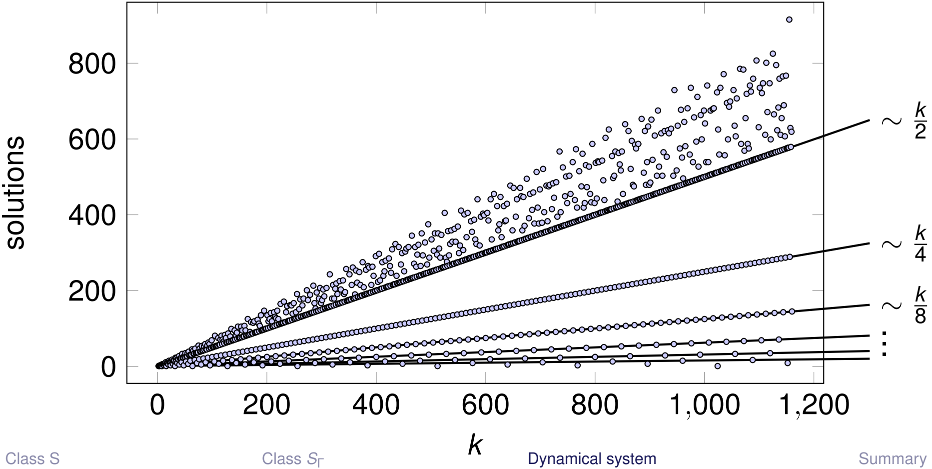
- even # of solutions has interesting structure



Summary

Even for the simplest class S_Γ theories, the punctures show an amazingly rich structure compared to the $\mathcal{N} = 2$ case.

still lots of questions

- quantitative measure for complexity
- connection to spin chain
- statistical properties of solutions
- are the characteristic quantities for a puncture
- can we do more for $N > 1$, e.g. large N limit AdS/CFT

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