Taking Advantage of Poisson-Lie Symmetry

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based on

18??.?????

with

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Motivation: A problem ...

- Task: Show that the WZ-model has $\mathcal{N} = 1$ SUSY
- very hard in components

  $S = \int dx^4 \left[\partial_\mu\phi\partial^\mu\phi^* + \overline{\psi}\not{\partial}\psi + \dots\right]$
  $\delta_\epsilon\phi = \bar{\epsilon}\psi$

  $\delta_\epsilon\psi = \not{\partial}\phi\epsilon + \dots$

- much simpler in superspace

  $S = \int dx^4 \int d^4\theta\Phi\Phi^\dagger - \int dx^4 \left[\int d\theta^2 W(\Phi) + \mathrm{h.c.}\right]$
  $\Phi = \phi + \theta\psi + \theta\theta F + \dots$

- we lean

  1. extension of spacetime with fermionic coordinates θ
  2. nonlinear realization of SUSY beomes linear
  3. component action after integrating out θ and aux. fields
  4. allows to derive non-renormalization theorem for $W(\Phi)$

Motivation: ...and a related problem

- Task: Check κ-symmetry of $AdS_3 \times S^3$ η-deformation
- very hard with (modified) SUGRA fields

$$ds = \frac{1}{(r^2-1)(1+r^2\kappa^2)} dr^2 + \frac{1+\rho^2}{(\kappa^2\rho^2-1)} dt^2 + \frac{(r^2+1)}{1+r^2\kappa^2} dx^2 + \frac{1}{(1+\rho^2)(1-\kappa^2\rho^2)} d\rho^2 + \dots$$

- much simpler in doubled space

$$\mathcal{H}^{AB} = \begin{pmatrix} \delta_{ab} & 0 \\ 0 & \delta^{ab} \end{pmatrix}, \qquad \begin{aligned} F_{abc} &= -\eta^{3/2} f_{ab}{}^{d} \delta_{dc} \\ F^{ab}{}_{c} &= \delta^{ad}\delta^{be}\delta_{cf} f_{de}{}^{f} \end{aligned}$$

- advantages
  1. (modified) SUGRA field equations become algebraic
  2. target space fields by contracting with gen. frame field
  3. dualities between integrable deformations are manifest
  4. naturally extents to the dilaton and the R/R sector

Outline

1. Motivation

2. Poisson-Lie T-duality/Symmetry

3. R/R sector of Double Field Theory on $\mathcal{D}$

4. Application to integrable deformations

5. Summary

Drinfeld double []

Definition: A **Drinfeld double** is a $2D$-dimensional Lie group $\mathcal{D}$, whose Lie-algebra $\mathfrak{d}$

1. has an ad-invariant bilinear for $\langle \cdot\,, \cdot \rangle$ with signature (D, D)
2. admits the decomposition into two maximal isotropic subalgebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$

- $\begin{pmatrix} t^a & t_a \end{pmatrix} = T_A \in \mathfrak{d}\,, \quad t_a \in \mathfrak{g}$ and $t^a \in \tilde{\mathfrak{g}}$
- $\langle T_A, T_B \rangle = \eta_{AB} = \begin{pmatrix} 0 & \delta^a_b \\ \delta_a^b & 0 \end{pmatrix}$
- $[T_A, T_B] = F_{AB}{}^C T_C$ with non-vanishing commutators

$$[t_a, t_b] = f_{ab}{}^c t_c \qquad [t_a, t^b] = \tilde{f}^{bc}{}_a t_c - f_{ac}{}^b t^c$$

$$[t^a, t^b] = \tilde{f}^{ab}{}_c t^c$$

- ad-invariance of $\langle \cdot\,, \cdot \rangle$ implies $F_{ABC} = F_{[ABC]}$

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- $[T_A, T_B] = F_{AB}{}^C T_C$ with non-vanishing commutators

$$[t_a, t_b] = f_{ab}{}^c t_c + f'_{abc} t^c \qquad [t_a, t^b] = \tilde{f}^{bc}{}_a t_c - f_{ac}{}^b t^c$$
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Poisson-Lie T-duality: 1. Definition [Klimčík and Severa, 1995]

- 2D σ-model on target space M with action

  $S(E, M) = \int dz d\bar{z}\, E_{ij} \partial x^i \bar{\partial} x^j$
- $E_{ij} = g_{ij} + B_{ij}$ captures metric and two-from field on M
- inverse of E_{ij} is denoted as E^{ij}
- *left* invariant vector field $v_a{}^i$ on G is the inverse transposed of *right* invariant Maurer-Cartan form $t_a v^a{}_i dx^i = -dg\, g^{-1}$
- adjoint action of $g \in G$ on $t_A \in \mathfrak{d}$: $\mathrm{Ad}_g\, t_A = g t_A g^{-1} = M_A{}^B t_B$
- analog for $\tilde{G}$

Definition: $S(E, \mathcal{D}/\tilde{G})$ and $S(\tilde{E}, \mathcal{D}/G)$ are **Poisson-Lie T-dual** if

$$E^{ij} = v_c{}^i M_a{}^c (M^{ae} M^b{}_e + S^{ab}) M_b{}^d v_d{}^j$$
$$\tilde{E}^{ij} = \tilde{v}^{ci} \tilde{M}^a{}_c (\tilde{M}_{ae} \tilde{M}_b{}^e + S_{ab}) \tilde{M}^b{}_d \tilde{v}^{dj}$$

holds, where S^{ab} is constant and invertible with the inverse S_{ab}.

Poisson-Lie T-duality: 2. Properties

- captures
  - abelian T-d. G abelian and $\tilde{G}$ abelian
  - non-abelian T-d. G non-abelian and $\tilde{G}$ abelian
  - [;; ;. . .]
- dual σ-models related by canonical transformation
  [Klimčík and Severa, 1995;Klimčík and Severa, 1996;Sfetsos, 1998]

$\rightarrow$ equivalent at the classical level

- preserves conformal invariance at one-loop
  [Alekseev, Klimcik, and Tseytlin, 1996;;. . . ;Jurco and Vysoky, 2018]
- Poisson-Lie symmetry: $L_{v_a} E_{ij} = -\tilde{f}^{bc}{}_a v^k{}_b v^l{}_c E_{ik} E_{lj}$
- η- , β- and λ^*-deformations admit Poisson-Lie symmetry

- What can we say about the R/R-sector?

Poisson-Lie T-duality: 2. Properties

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  - abelian T-d. — G abelian and $\tilde{G}$ abelian
  - non-abelian T-d. — G non-abelian and $\tilde{G}$ abelian
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2D σ-model perspective

(modified) SUGRA perspective

- What can we say about the R/R-sector?

O(D,D) Majorana-Weyl spinors on $\mathcal{D}$ [Hohm, Kwak, and Zwiebach, 2011,Hassler, 2018]

- Γ-matrices: $\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}$
- chirality Γ_{2D+1} with $\{\Gamma_{2D+1}, \Gamma_A\} = 0$
- charge conjugation C with $C\Gamma_A C^{-1} = (\Gamma_A)^\dagger$

- spinor can be expressed as $\chi = \sum_{p=0}^{D} \frac{1}{p!2^{p/2}} C^{(p)}_{a_1\dots a_p} \Gamma^{a_1\dots a_p}|0\rangle$
- Γ^a = creation op. and Γ_a = annihilation op. ($\{\Gamma^a, \Gamma_b\} = 2\delta^a_b$)
- $(\Gamma^a)^\dagger = \Gamma_a$ and $|0\rangle$ = vacuum ($\Gamma_a|0\rangle = 0$)
- χ is chiral/anti-chiral if all $C^{(p)}$ are even/odd

- O(D,D) transformation in spinor representation

$$\mathcal{S}_\mathcal{O}\Gamma_A\mathcal{S}_\mathcal{O}^{-1} = \Gamma_B\mathcal{O}^B{}_A \qquad \mathcal{O}^T\eta\mathcal{O} = \eta$$

R/R sector of DFT on $\mathcal{D}$ [Haßler, 2017]

- action $S_{\mathrm{RR}} = \frac{1}{4} \int d^{2d}X \, (\nabla\!\!\!/ \chi)^\dagger \, S_{\mathcal{H}} \, \nabla\!\!\!/ \chi$
- covariant derivative $\nabla\!\!\!/ \chi = \left(\Gamma^A D_A - \frac{1}{12}\Gamma^{ABC} F_{ABC} - \frac{1}{2}\Gamma^A F_A\right) \chi$
- flat derivative $D_A = E_A{}^I \partial_I$
- *left*-invariant vector fields $E_A{}^I$ constructed from *right*-invariant Maurer-Cartan form $T_A E^A{}_I = -\partial_I d d^{-1}$, $d \in \mathcal{D}$ as $E^A{}_I E_B{}^I = \delta^A_B$
- density part $F_A = D_A \log \left|\det(E^B{}_I)\right|$
- $\nabla\!\!\!/^2 = 0$ under SC (next slide)

- χ is chiral (IIB) or anti-chiral (IIA)
- satisfies self duality condition

$$G = -\mathcal{K} G \quad \text{with} \quad G = \nabla\!\!\!/ \chi \quad \text{and} \quad \mathcal{K} = C^{-1} S_{\mathcal{H}}$$

Symmetries of the action

- $S_{\mathrm{R/R}}$ invariant for $X^I \to X^I + \xi^A E_A{}^I$ and
  1. $\chi \to \chi + \mathcal{L}_\xi \chi$ and $\mathcal{H}^{AB} \to \mathcal{H}^{AB} + \mathcal{L}_\xi \mathcal{H}^{AB}$
  2. $\chi \to \chi + L_\xi \chi$ and $\mathcal{H}^{AB} \to \mathcal{H}^{AB} + L_\xi \mathcal{H}^{AB}$

1. generalized diffeomorphisms

$$\mathcal{L}_\xi \chi = \xi^A \nabla_A \chi + \frac{1}{2} \nabla_A \xi_B \Gamma^{AB} \chi + \frac{1}{2} \nabla_A \xi^A \chi$$

$$\mathcal{L}_\xi V^A = \xi^B \nabla_B V^A + \left(\nabla^A \xi_B - \nabla_B \xi^A \right) V^B + w \nabla_B \xi^B V^A$$

2. $2D$-diffeomorphisms

$$L_\xi \chi = \xi^A D_A \chi - \tfrac{1}{2}(\xi^A F_A - D_A \xi^A)\chi \quad \text{and} \quad L_\xi \mathcal{H}^{AB} = \xi^C D_C \mathcal{H}^{AB}$$

3. global O(D,D) transformations ($\mathcal{O}^A{}_C \mathcal{O}^B{}_D \eta^{CD} = \eta^{AB}$)

$$\chi \to S_{\mathcal{O}} \chi \quad \text{and} \quad \mathcal{H}^{AB} \to \mathcal{O}^A{}_C \mathcal{H}^{CD} \mathcal{O}^B{}_D$$

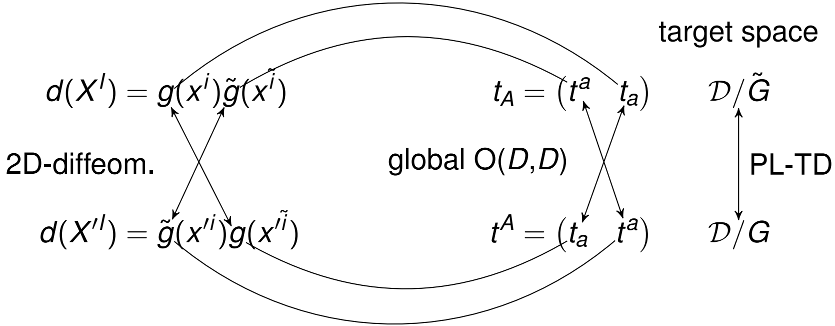
- section condition (SC) for f_1, f_2 with weights w_1, w_2

$$(D_A f_1 - w_1 F_A f_1)(D^A f_2 - w_2 F^A f_2) = 0$$

SC solutions and Poisson-Lie T-duality [Hassler, 2018;Haßler, 2017]

- fix D physical coordinates x^i from $X^I = \begin{pmatrix} x^i & x^{\tilde{i}} \end{pmatrix}$ on $\mathcal{D}$

  such that $\eta^{IJ} = E_A{}^I \eta^{AB} E_B{}^J = \begin{pmatrix} 0 & \cdots \\ \cdots & \cdots \end{pmatrix} \to$ SC is solved
- fields and gauge parameter depend just on x^i
- different SC solutions, relate them by symmetries of DFT



Equivalence to (m)SUGRA: 1. Generalized parallelizable spaces

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- generalized tangent space element $V^{\hat{I}} = \begin{pmatrix} V^i & V_i \end{pmatrix}$
- generalized Lie derivative

$$\widehat{\mathcal{L}}_\xi V^{\hat{I}} = \xi^{\hat{J}} \partial_{\hat{J}} V^{\hat{I}} + (\partial^{\hat{I}} \xi_{\hat{J}} - \partial_{\hat{J}} \xi^{\hat{I}}) V^{\hat{J}} \qquad \text{with} \qquad \partial_{\hat{I}} = \begin{pmatrix} 0 & \partial_i \end{pmatrix}$$

Definition: A manifold M which admits a globally defined generalized frame field $\widehat{E}_A{}^{\hat{I}}(x^i)$ satisfying

1. $\widehat{\mathcal{L}}_{\widehat{E}_A} \widehat{E}_B{}^{\hat{I}} = F_{AB}{}^C \widehat{E}_C{}^{\hat{I}}$

   where $F_{AB}{}^C$ are the structure constants of a Lie algebra $\mathfrak{h}$
2. $\widehat{E}_A{}^{\hat{I}} \eta^{AB} \widehat{E}_B{}^{\hat{J}} = \eta^{\hat{I}\hat{J}} = \begin{pmatrix} 0 & \delta^j_i \\ \delta^i_j & 0 \end{pmatrix}$

is a **generalized parallelizable space** $(M, \mathfrak{h}, \widehat{E}_A{}^{\hat{I}})$.

- SC solution on $\mathcal{D} \to$ gen. parallelizable space $(\mathcal{D}/\tilde{G}, \tilde{\mathfrak{g}}, \widehat{E}_A{}^{\hat{I}})$

Equivalence to (m)SUGRA: 2. R/R field strengths [Haßler, 2017]

see also [Y. Sakatani, S. Uehara, K. Yoshida, 2016; J. Sakamoto, Y. Sakatani, K. Yoshida, 2017]

- transport χ to the generalized tangent space:

$$\widehat{\chi} = |\det \tilde{e}_{ai}|^{-1/2} S_{\widehat{E}} \chi \qquad (t^a \tilde{e}_{ai} = \tilde{g}^{-1} d\tilde{g})$$

- same for covariant derivative

$$|\det \tilde{e}_{ai}|^{-1/2} S_{\widehat{E}} \nabla\!\!\!/ \chi = \left(\partial\!\!\!/ - \mathbf{X}_{\hat{I}} \widehat{\Gamma}^{\hat{I}} \right) \widehat{\chi} \quad \text{with} \quad \mathbf{X}_{\hat{I}} = \begin{pmatrix} I^i \\ -V_i \end{pmatrix}$$

$$S_{\widehat{E}} \Gamma^A S_{\widehat{E}}^{-1} \widehat{E}_A{}^{\hat{I}} = \widehat{\Gamma}^{\hat{I}} \quad \text{and} \quad \partial\!\!\!/ = \widehat{\Gamma}^i \partial_i$$

- $\mathbf{X}_{\hat{I}}$ vanishes if $\tilde{\mathfrak{g}}$ is unimodular
- introduce field strength $\widehat{F} = e^{\phi} S_B \left(\partial\!\!\!/ - \mathbf{X}_{\hat{I}} \widehat{\Gamma}^{\hat{I}} \right) \widehat{\chi}$
- and derivative $\mathbf{d} = e^{\phi} S_B \left(\partial\!\!\!/ - \mathbf{X}_{\hat{I}} \widehat{\Gamma}^{\hat{I}} \right) S_B^{-1} e^{-\phi}$

Equivalence to m(SUGRA): 3. field equations & BI

- DFT R/R field equations: $\not\nabla(\mathcal{K}\not\nabla)\chi = 0$
- rewrite them as:

  $\mathbf{d}(\star\mathbf{d}\widehat{F}) = 0 \quad \star = C^{-1}S_g^{-1}$
- puls Bianchi identity (BI)

  $\mathbf{d}\widehat{F} = 0$
- action on polyforms

  $\mathbf{d} \quad \leftrightarrow \quad d + H\wedge - Z\wedge - \iota_I$ with $Z = d\phi + \iota_I B - V$

  $\star \quad \leftrightarrow \quad \star$
- matches the R/R sector of (m)SUGRA [A. Tseytlin, L. Wulff, 2016]

Restrictions on $\mathcal{H}_{AB}$ and χ to admit Poisson-Lie Symmetry

- in general $\mathcal{H}_{AB}(x^i) \xrightarrow{\text{Poisson-Lie T-duality } (2D\text{-diff.})} \mathcal{H}_{AB}(x'^i, x'^{\tilde{i}})$
- $x'^{\tilde{i}}$ part not compatible with ansatz for SC solutions $\to$ avoid it

A doubled space $(\mathcal{D}, \mathcal{H}_{AB}, d)$ has Poisson-Lie symmetry iff

1. $L_\xi \mathcal{H}_{AB} = 0 \quad \forall \xi \quad \to \quad D_A \mathcal{H}_{BC} = 0$
2. $L_\xi \chi = 0 \quad \forall \xi \quad \to \quad D_A \chi = \frac{1}{2} F_A$

- BI for Poisson-Lie symmetric χ is algebraic

  $\nabla\!\!\!/ \chi = \frac{1}{12} F_{ABC} \Gamma^{ABC} \chi$
- finding R/R solutions reduces to linear algebra
- same holds NS/NS sector

  (here field equations are in general quadratic)

Application to integrable deformations

- starting point is solution to (m)CYBE

$$[\mathcal{R}x, \mathcal{R}y] - \mathcal{R}\left([\mathcal{R}x, y] + [x, \mathcal{R}y]\right) = -c^2[x, y]$$

- generalized metric after global O(D,D) very simple

$$\mathcal{H}^{AB} = \begin{pmatrix} \delta_{ab} & 0 \\ 0 & \delta^{ab} \end{pmatrix}$$

- structure coefficients have non-trivial components

$$F_{abc} = \kappa^{3/2} c^2 f_{ab}{}^d \delta_{dc} \,, \quad F_{ab}{}^c = 0 \,, \quad F^{ab}{}_c = \delta^{ad}\delta^{be}\delta_{cf} f_{de}{}^f \,, \quad F^{abc} = 0$$

- field equations for NS/NS + R/R sector become linear
- Poisson-Lie T-dualities between various deformations are manifest

Summary

- DFT, PL-Symmetry and integrable deformations fit together nicely
- interpretation of doubled space does not require winding modes anymore (phase space perspective instead)
- various interesting questions
  - implement coset spaces and dressing coset construction
  - fermionic sector and fermionic dualities
  - Drinfeld doubles $\rightarrow$ quantum groups $\rightarrow$ rich mathematical structure
  - new way to organized α' corrections?
  - new way to construct non-geometric backgrounds?
  - branes in curved space [Klimcik, and Severa, 1996 (D-branes)]?
- facilitates new applications
  - integrable deformations of 2D σ-models
  - solution generating technique
  - explore underlying structure of AdS/CFT

Additional structure on the Drinfeld double

[Blumenhagen, Hassler, and Lust, 2015, Blumenhagen, Bosque, Hassler, and Lust, 2015]

- *right* invariant vector $E_A{}^I$ field on $\mathcal{D}$ is the inverse transposed of *left* invariant Maurer-Cartan form $t_A E^A{}_I dX^I = g^{-1} dg$
- two η-compatible, covariant derivatives[1]
  1. flat derivative

     $$D_A V^B = E_A{}^I \partial_I V^B - w F_A V^B, \qquad F_A = D_A \log \left|\det(E^B{}_I)\right|$$

  2. convenient derivative

     $$\nabla_A V^B = D_A V^B + \tfrac{1}{3} F_{AC}{}^B V^C$$

- generalized metric $\mathcal{H}_{AB}$ ($w = 0$)

$$\mathcal{H}_{AB} = \mathcal{H}_{(AB)}, \qquad \mathcal{H}_{AC} \eta^{CD} H_{DB} = \eta_{AB}$$

- generalized dilaton d with e^{-2d} scalar density of weight $w = 1$
- triple $(\mathcal{D}, \mathcal{H}_{AB}, d)$ captures the doubled space of DFT

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[1] definitions here just for quantities with flat indices

Double Field Theory for $(\mathcal{D}, \mathcal{H}_{AB}, d)$ [Blumenhagen, Bosque, Hassler, and Lust, 2015]

see also [Vaisman, 2012; ;; ; . . .]

- action ($\nabla_A d = -\frac{1}{2} e^{2d} \nabla_A e^{-2d}$)

$$S_{\rm NS} = \int_{\mathcal{D}} d^{2D} X e^{-2d} \Big(\frac{1}{8} \mathcal{H}^{CD} \nabla_C \mathcal{H}_{AB} \nabla_D \mathcal{H}^{AB} - \frac{1}{2} \mathcal{H}^{AB} \nabla_B \mathcal{H}^{CD} \nabla_D \mathcal{H}_{AC} - 2 \nabla_A d \nabla_B \mathcal{H}^{AB} + 4 \mathcal{H}^{AB} \nabla_A d \nabla_B d + \frac{1}{6} F_{ACD} F_B{}^{CD} \mathcal{H}^{AB} \Big)$$

- $2D$-diffeomorphisms

$$L_\xi V^A = \xi^B D_B V^A + w D_B \xi^B V^A$$

- global O(D,D) transformations

$$V^A \to T^A{}_B V^B \quad \text{with} \quad T^A{}_C T^B{}_D \eta^{CD} = \eta^{AB}$$

- generalized diffeomorphisms

$$\mathcal{L}_\xi V^A = \xi^B \nabla_B V^A + \left(\nabla^A \xi_B - \nabla_B \xi^A \right) V^B + w \nabla_B \xi^B V^A$$

- section condition (SC)

$$\eta^{AB} D_A \cdot D_B \cdot = 0$$

Symmetries of the action

- S_{NS} invariant for $X^I \to X^I + \xi^A E_A{}^I$ and
  1. $\mathcal{H}^{AB} \to \mathcal{H}^{AB} + \mathcal{L}_\xi \mathcal{H}^{AB}$ and $e^{-2d} \to e^{-2d} + \mathcal{L}_\xi e^{-2d}$
  2. $\mathcal{H}^{AB} \to \mathcal{H}^{AB} + L_\xi \mathcal{H}^{AB}$ and $e^{-2d} \to e^{-2d} + L_\xi e^{-2d}$

object	gen.-diffeomorphisms	2D-diffeomorphisms	global O(D,D)
$\mathcal{H}_{AB}$	tensor	scalar	tensor
$\nabla_A d$	not covariant	scalar	1-form
e^{-2d}	scalar density (w=1)	scalar density (w=1)	invariant
η_{AB}	invariant	invariant	invariant
$F_{AB}{}^C$	invariant	invariant	tensor
$E_A{}^I$	invariant	vector	1-form
S_{NS}	invariant	invariant	invariant
SC	invariant	invariant	invariant
D_A	not covariant	covariant	covariant
∇_A	not covariant	covariant	covariant

manifest (2D-diffeomorphisms and global O(D,D))