

Last lecture: Three of four forces in nature are governed by Yang-Mills theory.

Knowing how to quantise it is paramount.

↳ Various challenges: 1. we are dealing with a gauge theory \rightarrow unphysical degrees of freedom

2. loop effects lead to ∞ 's which we have to regularise

\hookrightarrow 3. dependence on energy scale \rightarrow renormalisation

We will deal with them for the rest of the course.

Our most important tool = PATH INTEGRAL

5. Path Integral Formalism

5.1. Single Particle

Take a single particle moving in a potential $V(x)$ as warmup.

Hamiltonian: $H = \frac{P^2}{2m} + V(x)$

Question: What is the likelihood that the particle travels in the time T from point x_a to x_b ?

$$U(x_a, x_b; T) = \langle x_b | e^{-iHT/\hbar} | x_a \rangle$$

Basic idea behind the path integral: Write this amplitude as a sum over different phases along all paths from x_a to x_b .

$$U(x_a, x_b; T) = \sum_{\text{all paths}} e^{i \cdot \text{phase}} = \int Dx(t) e^{i \cdot \text{phase}}$$

"Sum" over ∞ many paths is written as integral over the continuous space of function $X(t)$.

$$\text{phase} = \frac{S[x(t)]}{\hbar}$$

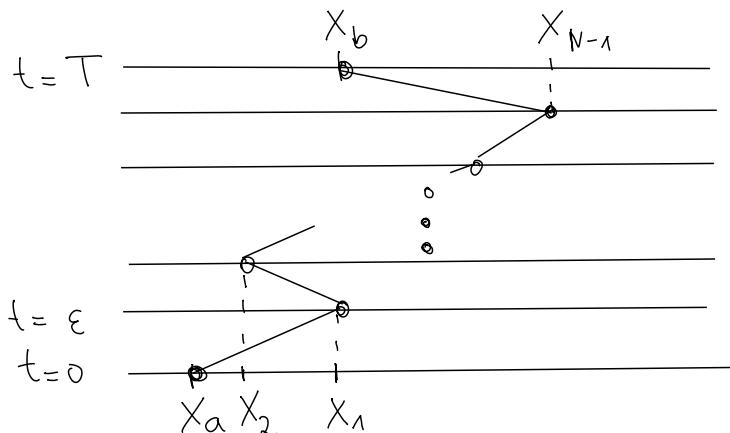
because then for $S \gg \hbar$ stationary phase approximation just gives the classical contribution $S[S[x(t)]] = 0$.

\hookrightarrow

$$U(x_a, x_b; T) = \int Dx(t) e^{i S[x(t)]/\hbar}$$

complicated integral
evaluate it by
discretisation

$$S = \int_0^T dt \left(\frac{m}{2} \dot{x}^2 - V(x) \right) \rightarrow \sum_K \left[\frac{m}{2} \frac{(x_{K+1} - x_K)^2}{\varepsilon} - \varepsilon V\left(\frac{x_{K+1} + x_K}{2}\right) \right]$$



$$\int Dx(t) = \int \frac{dx_{N-1}}{C(\varepsilon)}$$

⋮
⋮
⋮

$$\int \frac{dx_2}{C(\varepsilon)} \cdot \int \frac{dx_1}{C(\varepsilon)} \cdot \frac{1}{C(\varepsilon)}$$

constant we still have
to fix

$$\int Dx(t) = \frac{1}{C(\varepsilon)} \prod_K \int_{-\infty}^{\infty} \frac{dx_K}{C(\varepsilon)}$$

(I) Just the last step gives

$$U(x_a, x_b; T) = \int_{-\infty}^{\infty} \frac{dx'}{C(\varepsilon)} \exp \left[\frac{i}{\hbar} \frac{m(x_b - x')^2}{2\varepsilon} - \frac{i}{\hbar} \varepsilon V\left(\frac{x_b + x'}{2}\right) \right] \cdot U(x_a, x'; T - \varepsilon)$$

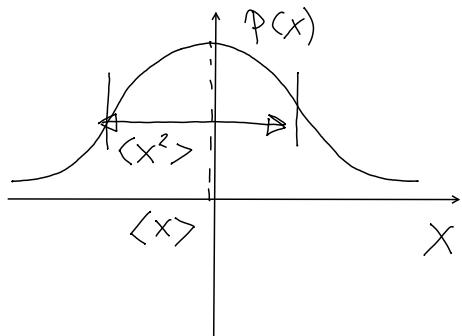
expand around $x' = x_b$

$$U(x_a, x_b; T) = \int_{-\infty}^{\infty} \frac{dx'}{C(\epsilon)} \exp\left(\frac{i}{\hbar} \frac{m}{2\epsilon} (x_b - x')^2\right) \left[1 - \frac{i\epsilon}{\hbar} V(x_b) + \dots \right]$$

$$\cdot \left[1 + (x' - x_b) \frac{\partial}{\partial x_b} + \frac{1}{2} (x' - x_b)^2 \frac{\partial^2}{\partial x_b^2} + \dots \right] U(x_a, x_b; T - \epsilon)$$

Gaussian integrals: $\int dx f(x) e^{-bx^2} = \langle f(x) \rangle$

$$\langle 1 \rangle = \sqrt{\frac{\pi}{b}}, \quad \langle x \rangle = 0, \quad \langle x^2 \rangle = \frac{1}{2b} \sqrt{\frac{\pi}{b}}$$



results in:

$$U(x_a, x_b; T) = \frac{1}{C(\epsilon)} \sqrt{\frac{2\pi\hbar\epsilon}{-im}} \cdot$$

$$\times \left(1 - \underbrace{\frac{i\epsilon}{\hbar} V(x_b)}_{\mathcal{E}H} + \frac{i\epsilon\hbar}{2m} \underbrace{\frac{\partial^2}{\partial x_b^2}}_{\mathcal{D}\mathcal{P}} + O(\epsilon^2) \right) U(x_a, x_b; T - \epsilon)$$

or

$$i\hbar \frac{\partial}{\partial T} U(x_a, x_b; T) = H U(x_a, x_b; T) \quad \text{Schrödinger equation}$$

This derivation is very powerful and also works for more complicated systems with more degrees of freedom.

$$U(\vec{q}_a, \vec{q}_b; T) = \left(\prod_i \int Dq_i(t) Dp_i(t) \right) \exp \left[i \int_0^T dt \left(\sum_i p_i \dot{q}_i - H(q_i, p_i) \right) \right]$$

integral over phase space

→ apply it to the Klein-Gordon field

5.2. Klein-Gordon Field

$$\text{remember: } H = \int d^3x \left[\frac{1}{2} \pi^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + V(\phi) \right]$$

$$\langle \phi_b(\vec{x}) | e^{-iHT} | \phi_a(\vec{x}) \rangle = \int D\phi D\pi \exp \left[i \int_0^T d^4x (\pi \dot{\phi} - H) \right]$$

quadratic in π , use Gaussian integral to compute $\int D\pi$

↗ EX 4

$$\langle \phi_b | e^{-iHT} | \phi_a \rangle = \int \mathcal{D}\phi \exp \left[i \int_0^T d^4x \mathcal{L} \right]$$

$$\text{with } \mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - V(\phi)$$

Let us now try to rederive the propagator

$$\langle 0 | T \hat{\phi}(x_1) \hat{\phi}(x_2) | 0 \rangle \text{ from the path integral.}$$

remember here $\hat{\phi}(x)$ are operators on the Hilbert space while in the path integral $\phi(x)$ are plain functions.

→ we look at the integral

$$\underbrace{\int \mathcal{D}\phi(x)}_{\int \mathcal{D}\phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x})} \phi(x_1) \phi(x_2) \exp \left[i \int_{-T}^T d^4x \mathcal{L}(\phi) \right]$$

$$\int \mathcal{D}\phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x}) \quad \begin{cases} \phi(x_1^\circ, \vec{x}) = \phi_1(\vec{x}) \\ \phi(x_2^\circ, \vec{x}) = \phi_2(\vec{x}) \end{cases}$$

time ordering is by definition build into the path integral and we find for $x_1^\circ < x_2^\circ$:

$$\begin{aligned} &= \int \mathcal{D}\phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x}) \phi(x_1^\circ) \phi(x_2^\circ) \langle \phi_b | e^{-iH(T-x_2^\circ)} | \phi_2 \rangle \\ &\quad \times \langle \phi_2 | e^{-iH(x_2^\circ - x_1^\circ)} | \phi_1 \rangle \langle \phi_1 | e^{-iH(x_1^\circ + T)} | \phi_a \rangle \\ &\quad + (x_1^\circ \leftrightarrow x_2^\circ) \text{ for } x_2^\circ < x_1^\circ \end{aligned}$$

Completeness relation: $\int \mathcal{D}\phi_1 | \phi_1 \rangle \langle \phi_1 | = 1$ (same for ϕ_2)

$$= \langle \phi_b | e^{-iHT} T [\hat{\phi}(x_1) \hat{\phi}(x_2)] e^{iHT} | \phi_a \rangle$$

remember $\hat{\phi}(x) = e^{iHx^\circ} \phi(\vec{x}) e^{-iHx^\circ}$

last step: $T \rightarrow \infty (1-i\varepsilon) \xrightarrow{\text{for convergence}}$

then $e^{-iHT} |\phi_a\rangle \sim |0\rangle \quad \left. \begin{array}{l} e^{iHT} |\phi_b\rangle \sim |0\rangle \end{array} \right\}$ all excitations
decay in the
infin't past /
future.

Success! We found the Feynman propagator from
the path integral.