

Example: ϕ^4 -theory

$$G^{(2)}(p) = \text{---} + \text{---} \bigcirc \text{---} + \text{---} \otimes \text{---} + \text{---} \bigcirc \text{---} + \dots$$

remember: wave function renormalisation
in ϕ^4 theory just @ two loops

$$G^{(4)} = \text{---} \times \text{---} + \text{---} \bigcirc \text{---} + \dots + \text{---} \otimes \text{---} + \mathcal{O}(\lambda^3)$$

s-, t- & u-channel

$$\dots = -i\lambda + (-i\lambda)^2 [iV(s) + iV(t) + iV(u)] - i\delta_\lambda$$

$$\delta_\lambda = (-i\lambda)^2 3V(-M^2) = \frac{3\lambda^2}{2(4\pi)^{d/2}} \int_0^1 dx \frac{\Gamma(2-d/2)}{(x(1-x)M^2)^{2-d/2}}$$

renormalisation condition $s = t = u = -M^2$

$$\hookrightarrow \delta_\lambda = \frac{3\lambda^2}{2(4\pi)^2} \left[\frac{1}{2-d/2} - \log M^2 + \text{finite} \right] \text{ for } m=0$$

We can now evaluate:

$$1) M \frac{\partial}{\partial M} G^{(4)} = M \frac{\partial}{\partial M} (-i\delta_\lambda) = \frac{3i\lambda^2}{(4\pi)^2} \text{ and}$$

$$2) \frac{\partial}{\partial \lambda} G^{(4)} = -i \text{ therefore}$$

$$\left[M \frac{\partial}{\partial M} + \beta(\lambda) \frac{\partial}{\partial \lambda} + 4\gamma(\lambda) \right] G^{(4)} = 0 \text{ implies}$$

$$\beta(\lambda) - 4i\gamma(\lambda) = \frac{3\lambda^2}{(4\pi)^2} + \mathcal{O}(\lambda^3)$$

$$\left[M \frac{\partial}{\partial M} + \beta(\lambda) \frac{\partial}{\partial \lambda} + 2\gamma(\lambda) \right] G^{(2)} = 0$$

first non-trivial contribution at two loops

$$\rightarrow \gamma(\lambda) = 0 + \mathcal{O}(\lambda) \text{ and } \beta(\lambda) = \frac{3\lambda^2}{(4\pi)^2} + \mathcal{O}(\lambda^3)$$

For a generic massless scalar theory:

$$G^{(2)}(p) = \text{---} + (\text{one loop diagrams}) + \text{---} \otimes \text{---} + \dots$$

$$= \frac{i}{p^2} + \frac{i}{p^2} (A \log \frac{\Lambda^2}{-p^2} + \text{finite}) + \frac{i}{p^2} (i p^2 \delta_2) \frac{i}{p^2} + \dots$$

at leading order $\mathcal{O}(\lambda)$ we can neglect $\beta(\lambda)$ and find

$$\gamma = \frac{1}{2} M \frac{\partial}{\partial M} \delta_2 = \boxed{-A = \gamma}$$

$$G^{(2)} = \left(\begin{array}{l} \text{tree-level} \\ \text{diagrams} \end{array} \right) + \left(\begin{array}{l} 1 \text{ PI loop} \\ \text{diagrams} \end{array} \right) + \left(\begin{array}{l} \text{Vertex} \\ \text{counter term} \end{array} \right) + \left(\begin{array}{l} \text{external leg} \\ \text{corrections} \end{array} \right)$$

$$= \left(\prod_i \frac{i}{p_i^2} \right) \left[-ig - iB \log \frac{\Lambda^2}{-p^2} - i\delta_g + (-ig) \sum_i (A_i \log \frac{\Lambda^2}{-p_i^2} - \delta_{z_i}) \right]$$

\nwarrow invariant build from p_i 's
 all $= -M^2$ by
 renormalisation condition

to lowest order we then find:

$$\beta(g) = M \frac{\partial}{\partial M} \left(-\delta_g + \frac{1}{2} g \sum_i \delta_{z_i} \right) = \boxed{-2B - g \sum_i A_i = \beta(g)}$$

Interpretation of $\gamma(\lambda)$ and $\beta(\lambda)$

remember: $\phi = Z(M)^{-1/2} \phi_0 \leftarrow$ bare
 renormalised \nearrow

$$\delta \eta = \frac{Z(M + \delta M)^{-1/2}}{Z(M)^{-1/2}} - 1 \Rightarrow \gamma(\lambda) = \frac{1}{2} M \frac{\partial}{\partial M} Z$$

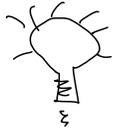
$$\text{Same argument for } \beta \Rightarrow \beta(\lambda) = M \frac{\partial}{\partial M} \lambda$$

rates of change of normalisation (γ) and couplings (β) when energy scale M changes

8.3 Evolution of the Coupling Constants

Original motivation: Integrate out one mass shell

after another. \leadsto renormalisation group flow



Callan-Symanzik equation = differential eq.
Solve (integrate) it!

$$G^{(n)}(p, \lambda) = \hat{G}^{(n)}(\bar{\lambda}(p; \lambda)) \cdot \exp\left(- \int_{p'=-M}^{p'=p} d \log\left(\frac{p'}{M}\right) \cdot n \left[1 - \gamma(\bar{\lambda}(p'; \lambda))\right]\right)$$

initial condition (measured at $p^2 = -M$)

$\bar{\lambda}$ = running couplings which solve:

$$\frac{d}{d \log\left(\frac{p}{M}\right)} \bar{\lambda}(p; \lambda) = \beta(\bar{\lambda}) \quad \text{and} \quad \bar{\lambda}(M; \lambda) = \lambda$$

$\beta(\bar{\lambda}) = 0$ is a fixed point of the flow