4.5. Root System

remember: last lecture Cartan subalgebra

generators $[H_i, H_j] = 0 \quad \forall H_i \in g_0, \; i = 1, \dots, \operatorname{rank} g$

Why? Because all element $H_i \in g_0$ can be diagonalised simultaniously

$$[H, Y] := ad_H(Y) = \alpha_Y(H)\, Y$$

eigen vector and corresponding eigen value

eigenvalues are $\boxed{\text{roots}}$ of the characteristic equation

They are assigned to each element of g_0:

$$\alpha_Y : g_0 \to \mathbb{C}$$

→ Roots are elements of the vector space g_0^* dual to g_0.

Idea: decompose Lie algebra g into

$$g = g_0 \bigoplus_{\alpha \neq 0} g_\alpha, \quad g_\alpha = \{ Y \in g \mid [H, Y] = \alpha(H)\, Y, \forall H \in g_0 \}$$

Cartan elements   non-zero roots

This is called root space decomposition of g.

g_0 = Cartan subalgebra & g_α = root subspaces

The collection of all the roots is called root system $\Phi = \Phi(g)$. For a simple Lie algebra, it has the properties:

1) The root system spans g_0^*, $\operatorname{span}_{\mathbb{C}}(\Phi) = g_0^*$.

↯ NOT a basis because more roots than rank g

2) For any $\alpha \in \Phi$, there is a $Y_{-\alpha} \in g$ such that

Killing metric $\longrightarrow$ $K(Y_\alpha, Y_{-\alpha}) \neq 0$.

3) The only multiples of $\alpha \in \Phi$ which are roots are $\pm\alpha$.

4) The root spaces g_α are one dimensional.

Homework: Try to prove 1) - 4).

Because of them we can introduce the

4.6. Cartan-Weyl Basis

I. basis for g_0 $\quad H_\alpha$ such that $\forall H \in g_0$

$$\alpha(H) := c_\alpha K(H_\alpha, H)$$

normalisation constants, we fix them later

This choice gives rise to the non-degenerate pairing

$$(\alpha, \beta) := c_\alpha c_\beta K(H_\alpha, H_\beta) = c_\alpha \beta(H_\alpha) = c_\beta \alpha(H_\beta)$$

on g_0^*.

II. for each $H_\alpha \in g_0$ there is an associated root E_α with $[H, E_\alpha] = \alpha(H) E_\alpha$.

$\rightarrow$ we can associate to any root α an $sl(2,\mathbb{C})$ subalgebra generated by $\{E_\alpha, E_{-\alpha}, H_\alpha\}$.

Check by calculating: $[E_\alpha, E_{-\alpha}]$

$$K(H, [E_\alpha, E_{-\alpha}]) = K([H, E_\alpha], E_{-\alpha})$$

ad-invariance of Killing form

$$= \alpha(H) K(E_\alpha, E_{-\alpha}) \neq 0 \text{ because of 2)}$$

$$= c_\alpha K(H, H_\alpha)$$

$K(\circ,\circ)$ is non-degenerate and eq. holds $\forall H \in \mathfrak{g}_0$

$\Rightarrow [E_\alpha, E_{-\alpha}] = c_\alpha K(E_\alpha, E_{-\alpha}) H_\alpha$, we also have:

$$[H_\alpha, E_{\pm\alpha}] = \pm\alpha(H_\alpha) E_{\pm\alpha} = \pm \underbrace{\frac{(\alpha,\alpha)}{c_\alpha}}_{} E_{\pm\alpha}$$

standard normalisation for $sl(2,\mathbb{C}) \to 2$

$$c_\alpha = \tfrac{1}{2}(\alpha,\alpha) \quad \text{and therefore} \quad \alpha(H_\beta) = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$$

We see that the structure of $sl(2,\mathbb{C})$ [= complexified version of $SU(2)$] occures multiple time inside a simple Lie algebra.

We already have $[H_\alpha, E_\beta]$ and $[E_\alpha, E_{-\alpha}]$, but what about $[E_\alpha, E_\beta]$?

Tool root vector: $\quad [H_i, E_\alpha] = \alpha_i E_\alpha$

or root for short $\quad \alpha_i := \alpha(H_i)$

$$\begin{aligned}[H_i,[E_\alpha,E_\beta]] &= -[E_\alpha,[E_\beta,H_i]] - [E_\beta,[H_i,E_\alpha]] \\ &= [E_\alpha,[H_i,E_\beta]] + [[H_i,E_\alpha],E_\beta] \\ &= (\alpha_i+\beta_i)[E_\alpha,E_\beta]\end{aligned}$$

For $\alpha+\beta \neq 0$ this implies that $[E_\alpha, E_\beta]$ is proportional to the generator $E_{\alpha+\beta}$ of the root subspace $\mathfrak{g}_{\alpha+\beta}$, provided that $\alpha+\beta \in \Phi$. To summarise:

Cartan-Weyl basis

$$\boxed{\begin{aligned} &[H_i, E_\alpha] = \alpha_i E_\alpha, \qquad [H_i, H_j] = 0 \\ &[E_\alpha, E_{-\alpha}] = H_\alpha = \alpha^\vee_i H^i, \qquad [E_\alpha, E_\beta] = \begin{cases} e_{\alpha,\beta} E_{\alpha+\beta}, & \alpha+\beta \in \Phi \\ 0, & 0 \neq \alpha+\beta \notin \Phi \end{cases} \end{aligned}}$$

co-root for α

with the normalisation $K(E_\alpha, E_{-\alpha})\, c_\alpha = 1$
which implies $K(E_\alpha, E_\beta) = \frac{2}{(\alpha,\alpha)}\, \delta_{\alpha,-\beta}$.

4.7. Example $sl(3,\mathbb{C})$

interesting for physics because complexification of $SU(3) \sim$ gauge group of QCD.

generators are traceless, real 3×3 matrices; in total

$$\underbrace{2 \text{ diagonal}}_{\text{Cartan subalgebra}} + \underbrace{3 \text{ upper triangular} + 3 \text{ lower tri}}_{\text{non-zero roots}} = 8$$

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Killing form: $K(X, Y) = \mathrm{Tr}(X \cdot Y)$

We take $c_\alpha = 1$ and therefore $(H_i, H_j) = \begin{pmatrix} \boxed{\begin{matrix} 2 \\ -1 \end{matrix}} & \boxed{\begin{matrix} -1 \\ 2 \end{matrix}} \end{pmatrix}$

taking $E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

we find:
$$[H_1, E_1] = \boxed{2}\, E_1 \qquad [H_1, E_2] = \boxed{-1}\, E_2$$
$$[H_2, E_1] = \boxed{-1}\, E_1 \qquad [H_2, E_2] = \boxed{2}\, E_2$$

$E_{-1} = E_1^T$, $E_{-2} = E_2^T$ check $K(E_\alpha, E_{-\beta}) = \delta_{\alpha,\beta}$

finally $E_3 = [E_1, E_2] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $E_{-3} = E_3^T$

Homework check all the other relations