An Introduction to String Theory, Winter 2022/23

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5. String solutions and gravitational waves (18 points)

To be discussed on Thursday, 3rd November, 2022 in the tutorial.
Please indicate your preferences until Saturday, 29/10/2022, 21:00:00 on the website.

Exercise 5.1: Classical string solutions

We showed during the lecture that in conformal gauge any classical string solution is of the form

$$X^\mu(\tau,\sigma) = \frac{1}{2}\left(F^\mu(\tau+\sigma) + G^\mu(\tau-\sigma)\right)\,, \tag{1}$$

where the function $F^\mu(u)$ and $G^\mu(v)$ are valued in D-dimensional Minkowski space. In this problem, we only consider closed strings, implying that $X^\mu(\tau,\sigma+2\pi) = X^\mu(\tau,\sigma)$ holds.

a) (1 point) Additionally, we know that the motion of the string is constraint by

$$\left(\dot{X} \pm X'\right)^2 = 0\,.$$

Remember where these two constraints come form and show how they affect $F^\mu(u)$ and $G^\mu(u)$?

b) (2 points) Fix

$$X^0(\tau,\sigma) = \tau$$

and rewrite the general solution in terms of two new, periodic functions $f^i(u)$ and $g^i(u)$ valued in the $(D-2)$-dimensional unit sphere, and the appropriate integration constants.

Hint: You might get a better intuition how this works, by starting with the simplest example in $D = 3$ where the unit sphere is just a unit circle.

c) (2 points) Write down the time evolution of a string which, at time $X^0 = 0$, forms a circle of radius R at rest in the X^1-X^2-plane.

d) (2 points) Calculate the mass of the solution found in c).

e) (2 points) Show that, for a generic solution in $D = 4$, there are points u_* and v_* in the parameter space for which $f(u_*) = g(v_*)$.

f) (3 points) Show that around such points, the trace of the string in spacetime forms a cusp singularity, moving (instantaneously) at the speed of light.

Hint: An (ordinary) cusp in the x-y-plane can be defined (locally) as the set of solutions of the equation $x^3 = y^2$. You will find the cusp in parameterised form by expanding f and g around the singular point.

Exercise 5.2: Gravitational waves

In the last exercise, we encountered the field equations of general relativity, also called Einstein equations. In D-dimensional spacetime, they read

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \frac{\kappa_D^2}{4}T_{\mu\nu} \tag{2}$$

for some constant κ_D. For many physical phenomena gravity is very weak, and the metric $g_{\mu\nu}(x)$ can be chosen to be very close to the Minkowski metric

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa_D h_{\mu\nu}(x)$$

by viewing $h_{\mu\nu}$ as a small fluctuation.

a) (2 points) Show that to the first order in $h_{\mu\nu}$ the Einstein equations (2) take the form

$$\partial^\sigma \partial_\sigma (h_{\mu\nu} - \eta_{\mu\nu} h^\rho{}_\rho) - \partial^\sigma (\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu}) + \partial_\mu \partial_\nu h^\sigma{}_\sigma + \eta_{\mu\nu} \partial^\rho \partial^\lambda h_{\rho\lambda} = -\frac{\kappa_D}{2} T_{\mu\nu} \qquad (3)$$

where indices are raised and lowered with $\eta^{\alpha\beta}$ and $\eta_{\alpha\beta}$.
Hint: Here you clearly have to compute an infinitesimal, because $h_{\mu\nu}$ is very small, variation of (2). You can find, i.e. on the Wikipedia page for the Einstein-Hilbert action how this is done for its individual components (see the section "2.2 Variation of the Riemann tensor, Ricci tensor and the Ricci scalar")

b) (1 point) Show that by introducing

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h^\sigma{}_\sigma \eta_{\mu\nu}$$

you can bring (3) to the form

$$\partial^\sigma \partial_\sigma \overline{h}_{\mu\nu} - \partial^\sigma \left(\partial_\mu \overline{h}_{\sigma\nu} + \partial_\nu \overline{h}_{\sigma\mu} \right) + \eta_{\mu\nu} \partial^\rho \partial^\lambda \overline{h}_{\rho\lambda} = -\frac{\kappa_D}{2} T_{\mu\nu} \,.$$

c) (1 point) Explain from what we learned in the lecture and discussed in the last exercise why the metric $g_{\mu\nu}$ transforms like

$$\delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$

under diffeomorphisms.

d) (2 points) Consider a diffeomorphism that solves the inhomogeneous Laplace equation

$$\partial^\sigma \partial_\sigma \xi_\mu = -\partial^\nu h_{\nu\mu} + \frac{1}{2} \partial_\mu h^\sigma{}_\sigma$$

Show that after applying it, the linearised Einstein equations reduce to

$$\partial^\sigma \partial_\sigma \overline{h}'_{\mu\nu} = \frac{\kappa}{2} T_{\mu\nu}$$

(with $\overline{h}'_{\mu\nu}$ arising after the diffeomorphism).
In the absence of a source this is just a wave equation describing gravitational waves.