

8. Representations

In physics we don't see Lie algebra directly, but only its consequences \rightarrow representations

i.e. $[t_a, t_b] = f_{ab}^c t_c$

task: map t_a, t_b and $[.,.]$ to known structures like matrices A, B and $[A, B] = A \cdot B - B \cdot A$ matrix multipl. \rightarrow

8.1. Module

Def.: Take a Lie algebra \mathfrak{g} and a linear vector space V over a field \mathbb{F} . A representation ρ of \mathfrak{g} is a linear mapping:

$$\rho(x): V \rightarrow V \quad \text{for } \forall x \in \mathfrak{g}$$

such that: $\boxed{\rho(x) \circ \rho(y) - \rho(y) \circ \rho(x) = \rho([x, y])}$

remarks: - \circ is the usual composition i.e.

$$\rho(x) \circ \rho(y)[v] = \rho(x)[\rho(y)[v]], \quad v \in V$$

- V is a vector space & ρ is linear $\rightarrow \rho: \mathfrak{g} \rightarrow \text{gl}(n, \mathbb{F})$

$\rightarrow \rho(x)$ can be thought of as $n \times n$ matrix over the field \mathbb{F} \leftarrow dimension of V

$\rho: \mathfrak{g} \rightarrow \text{gl}(V)$ is an homomorphism of \mathfrak{g} structure preserving map \rightarrow

V is called the representation space or module (V, ρ)

A module is also equipped with another operation

$$\bullet : \mathfrak{g} \times V \rightarrow V$$

$(x, v) \mapsto x \bullet v = w \in V$

\uparrow not the composition?

such that for $a, b \in \mathbb{F}$, $x, y \in \mathfrak{g}$ and $v, w \in V$

$$1) (ax + by) \cdot v = a(x \cdot v) + b(y \cdot v),$$

$$2) x \cdot (av + bw) = a(x \cdot v) + b(x \cdot w),$$

$$3) [x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v) \quad \text{hold}$$

$\underbrace{x \cdot v}_{\text{module}} = \underbrace{\rho(x)[v]}_{\text{representation}}$
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\Rightarrow in physics literature a module is often called a representation.

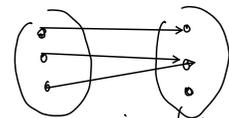
Properties of modules:

I) two modules (V, \mathfrak{g}) and (W, \mathfrak{g}) for the same Lie algebra \mathfrak{g} and a linear mapping $f: V \rightarrow W$ such that

$$\rho_W(x) \circ f = f \circ \rho_V(x)$$

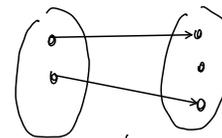
f intertwines the action of \mathfrak{g} and is called an intertwiner

II) If f is surjective, W is called the homomorphic image of V .



surjective and

III) If f is also injective, then (V, \mathfrak{g}) and (W, \mathfrak{g}) are isomorphic modules.



injective

They are indistinguishable.

one-to-one

IV) A representation is called faithful or effective if $x, y \in \mathfrak{g}$ with $x \neq y \rightarrow \rho(x) \neq \rho(y)$.

8.2 Constructing representations

Starting from a single representation ρ of \mathfrak{g} acting on the module V , one can generate other representations by the following operations from linear algebra:

I) conjugation: $\bar{\rho}(x) := -\rho^T(x)$

works because $[t_a, t_b]^T = [t_b^T, t_a^T] = -[t_a^T, t_b^T]$

$$-(fab^c t_c)^T = fab^c t_c^T$$

$$\hookrightarrow [-t_a^T, -t_b^T] = fab^c (-t_c^T) //$$

This representation acts on the dual vector space V^* .

II) direct sum: Take two \mathfrak{g} modules, V and W , then we can construct the direct sum module $V \oplus W$ with:

$$(\rho_V \oplus \rho_W)(x)[V \oplus W] := \rho_V(x)[V] \oplus \rho_W(x)[W]$$

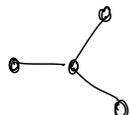
III) tensor product:

$$(\rho_V \otimes \rho_W)(x)[V \otimes W] := \rho_V(x)[V] \otimes W + V \otimes \rho_W(x)[W]$$

IV) endomorphism: For an homomorphism of the Lie algebra $\varphi: \mathfrak{g} \rightarrow \mathfrak{g}$ we can construct

$$\rho^\varphi(x) = \rho(\varphi(x))$$

preserves brackets

Example: $so(8)$  has three fold symmetry

\hookrightarrow three 8 dimensional representations

$\rho_s, \rho_c, \rho_v \leftarrow$ vector called triality of $so(8)$
 \uparrow spinor \uparrow conjugate spinor

8.3 Irreducible representations

These constructions lead to many representations. But they might decompose into smaller representations.

Remember: 1st lecture product representation of 3-dimensional representation of $su(2)$

$$3 \otimes 3 \rightarrow 1 \oplus 3 \oplus 5$$

Question: What are the smallest representations one can get from a given representation?

Answer: Irreducible representations = irreps

Def.: A submodule U of a module V is a subvector-space $U \subset V$ such that:

$$\rho(x)[U] \subseteq U \quad \text{for } \forall x \in \mathfrak{g}.$$

Each module has two trivial submodules:

- the module itself and
- the zero vector space $\{0\}$.

An irrep does not possess any other submodules than these two trivial ones.

\Rightarrow An reducible representation contains non-trivial submodules.

Schur's Lemma: Any intertwiner f between two irreducible modules V and W is either an isomorphism or zero ($f = 0$).

A module is called fully reducible if it can be written as a direct sum of irreps.

Any Lie algebra has always the following three irreps:

a) trivial representation: $\rho(x) = 0$

But we can't learn much about \mathfrak{g} from it.

b) adjoint representation: $\text{ad}_x(y) = [x, y], x, y \in \mathfrak{g}$

Here $V = \mathfrak{g}$ and we have used it extensively before.

c) fundamental representation:

From it all other reps can be obtained with the tools from section 8.2.