9. su(N) representations

remember:

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & & su(N) \text{ — today!} \\
 & & \boxed{\text{classical}} \arrow[ur] \arrow[r] \arrow[dr] & so(N) \\
 & \boxed{\text{simple}} \arrow[ur] \arrow[dr] & & sp(2N) \text{ — next week} \\
\boxed{\text{Lie algebras}} \arrow[ur] \arrow[r] \arrow[dr] & \text{solvable} & \boxed{\text{exceptional}} & \dots \\
 & \text{nilpotent} & &
\end{tikzcd}$$

The Lie group $SU(N)$ is defined by its action on a N-component vector v with complex components v^i:

$$M: v \mapsto v' = M \cdot v \quad , \quad v^i \mapsto v'^i = M^i{}_j v^j ,$$

with $M^\dagger M = \mathbb{1}$ and $\det M = 1$.

⇝ For the Lie algebra $su(N)$ we have: $X \in su(N)$

$$X: \quad v \mapsto v' = \rho(x)\, v = X v$$
$$v^i \mapsto v'^i = X^i{}_j v^j ,$$

with $X^\dagger + X = 0$ and $\operatorname{tr} X = 0$

remarks:
- fundamental representation of $su(N)$
- $V = \mathbb{C}^N$ is the fundamental module
- also called vector representation $\hat{=}$ (N)

We usually distinguish irreps by their dimension. If there are several with the same dim., we add decoration like $\bar{\ }$, $'$ ...

last lecture: all other irreps can be derived from fundamental

A) I.e. take $v \otimes w \in (N) \otimes (N)$

and decompose into (anti-) symmetric part ↗ 3.1. $gl(N)$ reps.

$$(v \vee w)^{ij} = \frac{1}{2}\left(v^i \otimes w^j + v^j \otimes w^i\right) \quad \text{symmetric}$$

$$(v \wedge w)^{ij} = \frac{1}{2}\left(v^i \otimes w^j - v^j \otimes w^i\right) \quad \text{anti-symmetric}$$

both define irreps and we write:

$$(N) \otimes (N) = \underbrace{\left(\tfrac{1}{2} N(N+1)\right)}_{\text{sym.}} \oplus \underbrace{\left(\tfrac{1}{2} N(N-1)\right)}_{\text{anti-sym.}}$$

B) conjugation

i.e. of the fundamental $(N) \to (\bar{N})$

(N): $v \mapsto X \cdot v$

$(\bar{N})$: $\bar{v} \mapsto \bar{v}(-X)$, $\quad \bar{v}_i \mapsto \bar{v}_j (-X)^j{}_i$

last lecture we had: $\quad w \mapsto (-X^T)\, w$

$\quad \rightsquigarrow \quad \bar{v} := w^T$ //

A + B) $\quad (\bar{N}) \otimes (\bar{N}) = \left(\overline{\tfrac{1}{2} N(N+1)}\right) \oplus \left(\overline{\tfrac{1}{2} N(N-1)}\right)$

or $(N) \otimes (\bar{N})$: $\quad t^i{}_j \mapsto \sum_k X^i{}_k t^k{}_j + \sum_e (-X)^e{}_j t^i{}_e$

where we decompose: $\quad t^i{}_j = \frac{1}{N} s\, \delta^i{}_j + ad^i{}_j$

with

$$s := tr(t) = \sum_k t^k{}_k \quad \text{and} \quad ad^i{}_j := t^i{}_j - \frac{1}{N} tr(t)\, \delta^i{}_j$$

- s is called singlet because the module is one (= single) dimensional = trivial representation.

$$s = \sum_i t^i{}_i \mapsto s' = \sum_j \left(\sum_k X^j{}_k t^k{}_j + \sum_e (-X)^e{}_j t^j{}_e \right)$$
$$= \sum_{j,k} \left(X^j{}_k t^k{}_j - X^j{}_k t^k{}_j \right) = 0$$

- $ad^i{}_j$ is the adjoint module (please check) with $\dim(ad) = N^2 - 1$

Note: $\delta^i{}_j$ is an invariant tensor of $su(N)$.
It does not transform. $\rightsquigarrow$ used for singlet

The second invariant tensor of $su(N)$ is called Levi-Civita tensor $\varepsilon_{i_1 \dots i_N}$

It gives rise to the singlet $(N)^{\wedge N} = \underbrace{(N)\wedge\ldots\wedge(N)}_{N\text{-times}} = (1)$

$\hat{=}\quad v^{i_1}\wedge\ldots\wedge v^{i_N} = s\,\varepsilon^{i_1\ldots i_N}.$

$\rightarrow$ $\boxed{\text{su}(N)\ N\text{-times totally anti-symm. irrep} = \text{trivial irrep}}$

In a similar spirit we find:

$$(N)^{\wedge(N-1)} = \overbrace{(N)\wedge\ldots\wedge(N)}^{N-1} = (\bar{N})$$

$$v^{i_1}\wedge\ldots\wedge v^{i_{N-1}} = \varepsilon^{i_1\ldots i_{N-1}j}\,\bar{v}_j$$

$$\bar{v}_j = \frac{1}{(N-1)!}\,\varepsilon_{i_1\ldots i_{N-1}j}\,v^{i_1}\ldots v^{i_{N-1}}$$

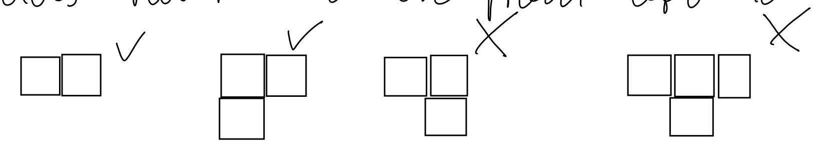
8.1. Young tableaux

We have seen: (anti-) symmetrising in tensor products gives rise to irreps!

Idea: diagrammatic representation

Rules: • A rank n tensor $t^{i_1\ldots i_n}$ is represented by n boxes.

• Draw them in columns such that their number does never increase from left to right.



• anti-symmetrise over columns & symmetrise over rows:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} \;\hat{=}\; \big(()-(13)\big)\big(()+(12)\big) = () + (12) - (13) - (132)$$

$$t'^{\,i_1 i_2 i_3} = t^{i_1 i_2 i_3} + t^{i_2 i_1 i_3} - t^{i_3 i_2 i_1} - t^{i_3 i_1 i_2}$$

with $t'^{\,i_1 i_2 i_3} = t'^{\,i_2 i_1 i_3} = -t'^{\,i_3 i_2 i_1}$

$\boxed{\text{For su}(N)\text{ at most } N \text{ boxes in column!}}$

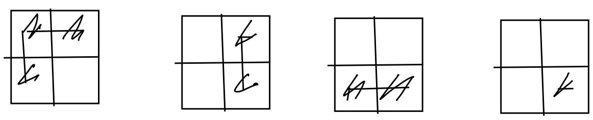
$\square = (N)$, $N\left\{\begin{array}{c}\square\\ \vdots\\ \square\end{array}\right. = (1)$, $N-1\left\{\begin{array}{c}\square\\ \vdots\\ \square\end{array}\right. = (\bar{N})$

irrep ~ different Young tableaux

- dim. of corresponding irrep:

$$\dim = \frac{\text{numerator}}{\text{denominator}}$$

multiply hook length for each cell:

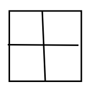


$$3 \cdot 2 \cdot 2 \cdot 1 = 12$$

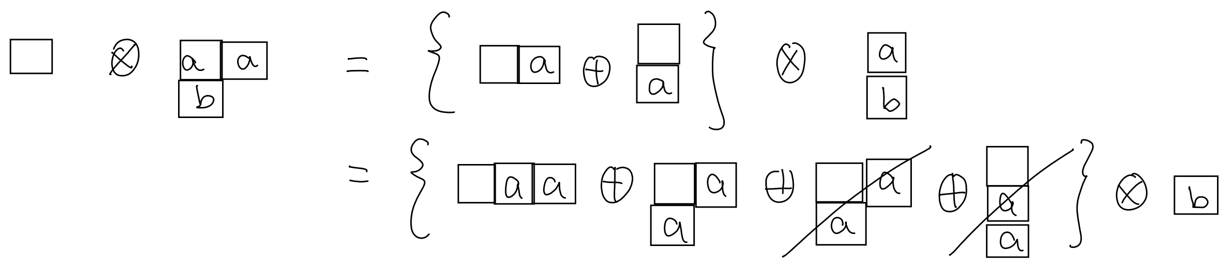
- start with N in the top left corner.
- Add one when you go $\rightarrow$, substract one when you go $\downarrow$

N	$N+1$
$N-1$	N

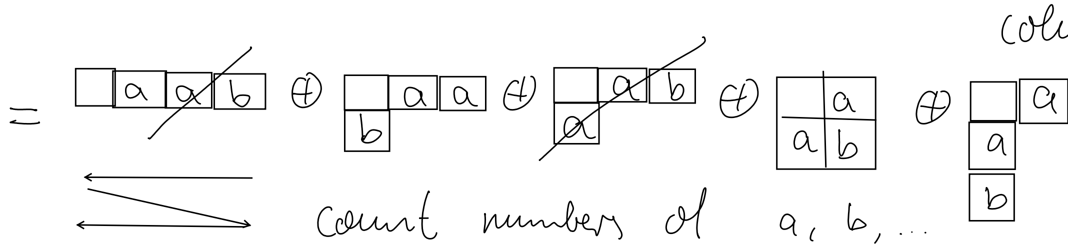
- multiply $= N^2(N^2-1)$

dim() $= \frac{1}{12} N^2 (N^2-1)$

- decomposition of tensor products:



duplicate two a's in column



count numbers of a, b, ...
if they are not strictly increase delete the tableaux