An Introduction to String Theory, Winter 2022/23

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9. Conformal Field Theory (20 points)

To be discussed on Thursday, 15th December, 2022 in the tutorial.
Please indicate your preferences until Saturday, 10/12/2022, 21:00:00 on the website.

Exercise 9.1: Conformal transformations as angle preserving maps

In general, a conformal transformation $x \to \tilde{x}(x)$ is defined to be a transformation that preserves the metric up to a local scale factor,

$$\tilde{g}_{pq}(\tilde{x}(x)) = \Omega^2(x) \frac{\partial x^m}{\partial \tilde{x}^p} \frac{\partial x^n}{\partial \tilde{x}^q} g_{mn}(x) \,.$$

Specializing from now on to positive definite curved metrics (i.e. with Euclidean signature) the angle α between two vector fields $v^m(x)$ and $w^m(x)$ at a point x_0 is defined by

$$\cos\alpha(v,w)(x_0) := \left. \frac{v^m w^n g_{mn}}{||v|| \, ||w||} \right|_{x=x_0} \,.$$

Here, $||v|| := \sqrt{v^m v^n g_{mn}}$ is the length, or norm, of the vector v^m.

a) (2 points) Show that a conformal transformation is angle-preserving, i.e., that

$$\cos\alpha(\tilde{v},\tilde{w})(\tilde{x}(x_0)) = \cos\alpha(v,w)(x_0) \,,$$

where

$$\tilde{v}^m(\tilde{x}(x)) = \frac{\partial \tilde{x}^m}{\partial x^n} v^n(x)$$

is the transformed vector field.

b) (1 point) In conformal gauge, the 2D Lorentzian world sheet metric is

$$\begin{aligned} ds^2 &= \Omega^2(\sigma,\tau)(-d\tau^2 + d\sigma^2) \\ &= -\Omega^2(\sigma^+,\sigma^-) d\sigma^+ d\sigma^- \,. \end{aligned}$$

Performing the Wick rotation

$$\sigma^\pm = (\tau \pm \sigma) \to -i(\tau \pm i\sigma)$$

and write down the resulting Euclidean metric both in terms of the (Wick-rotated) (τ,σ) and the complex coordinates

$$z' = \tau - i\sigma \,, \quad \bar{z}' = \tau + i\sigma \,.$$

c) (2 points) Show that all holomorphic coordinate transformations

$$z' \to \tilde{z}(z') \,, \quad \bar{z}' \to \bar{\tilde{z}}(\bar{z}')$$

change the metric only by a local rescaling $\Omega^2(z',\bar{z}') \to f(z',\bar{z}')\Omega^2(z',\bar{z}')$ or equivalently that they are conformal.

Exercise 9.2: Fractional linear transformation

Remember from exercise 4.3 d) that the generators L_{-1}, L_0 and L_1 form a subalgebra of the full Virasoro algebra. They generate the Lie group $\mathrm{SL}(2,\mathbb{R})$, which consists of real 2×2-matrices with unit determinant. Here, we want to see how this group acts on the Riemann-sphere $\mathbb{C}\cup\{\infty\}$ by so-called fractional linear transformations,

$$z \to z' = \frac{az+b}{cz+d}\,,$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})\,. \tag{1}$$

a) (2 points) Show that two successive fractional linear transformations,

$$z \to z' = \frac{az+b}{cz+d}\,, \quad z' \to z'' = \frac{ez'+f}{gz'+h}\,,$$

are equivalent to one fractional linear transformation

$$z \to z'' = \frac{jz+k}{lz+m}\,,$$

where the matrix

$$\begin{pmatrix} j & k \\ l & m \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$$

is the product of two $\mathrm{SL}(2,\mathbb{R})$ matrices that correspond to the single transformations $z \to z'$ and $z' \to z''$.

b) (2 points) Show that the fractional linear action of the inverse matrix of (1) on z' leads back to z, and hence corresponds to the inverse transformation $z' \to z$.

Exercise 9.3: Correlation functions for the bosonic string

Consider the CFT of D free bosons X^μ on the plane that is governed by the action

$$S = \frac{1}{4\pi\kappa}\int \mathrm{d}z\mathrm{d}\bar{z}\,\partial X^\mu \bar{\partial} X_\mu\,. \tag{2}$$

a) (1 point) Show that this action result in the equation of motion

$$\partial\bar{\partial}X^\mu(z,\bar{z}) = 0\,.$$

b) (1 point) Check that the theory given by (2) is invariant under conformal transformations of $X^\mu(z,\bar{z})$ with the conformal dimensions h=0 and $\bar{h}$=0.

c) (1 point) Verify that the propagator of this theory is

$$\langle X^\mu(z,\bar{z})X^\nu(w,\bar{w})\rangle = -\kappa\log|z-w|^2\eta^{\mu\nu}\,. \tag{3}$$

Hint: Remember that the propagator is the Green's function

$$\partial\bar{\partial}\langle X^\mu(z,\bar{z})X^\nu(w,\bar{w})\rangle = -2\pi\kappa\delta^{(2)}(z-w)\,.$$

Using the representation of the δ-function $2\pi\delta^{(2)}(z) = \partial\bar{z}^{-1}$ you should show that the propagator given in (3) solves this equation.

For a free theory, the propagator is all we need to know to compute any correlation function with the help of Wick's theorem. So let us compute the following three correlators:

d) (2 points) $\langle X^\mu(z_1,\bar{z}_1)X_\mu(z_2,\bar{z}_2)X^\nu(z_3,\bar{z}_3)X_\nu(z_4,\bar{z}_4)\rangle$,

e) (1 point) $\langle \partial X^\mu(z_1,\bar{z}_1)\partial X^\nu(z_2,\bar{z}_2)\rangle$, and

f) (1 point) $\langle \partial X^\mu(z_1,\bar{z}_1)\bar{\partial} X^\nu(z_2,\bar{z}_2)\rangle$.

Exercise 9.4: Operator product expansion

Follow the lecture to compute the OPE of

a) (2 points) $\partial X^\mu(z)\partial X^\nu(0)$ and

b) (2 points) $\partial X^\mu(z)\bar{\partial} X^\nu(0)$.

Hint: All you need is the mode expansion of ∂X^μ and $\bar{\partial} X^\mu$. The modes are proportional to our good old friends α^μ_m and $\bar{\alpha}^\mu_m$. For them you know the commutators. That and the lecture notes is all you need to know to compute the two OPE that are asked for.