

remember: last time superficial degree of divergence D in four dimensions

in d dimensions we find

$$D = dL - P_e - 2P_f = \dots \nearrow \boxed{\text{EX 9.6}}$$

$$= d + \left(\frac{d-4}{2}\right)V - \left(\frac{d-2}{2}\right)M_f - \left(\frac{d-1}{2}\right)M_e$$

three options:

- $d < 4$ super-renormalisable: Only a finite number of Feynman diagrams superficially diverge.
- $d = 4$ renormalisable: Only a finite number of amplitudes sup. diverge; but divergences occur at all orders in perturbation theory.
- $d > 4$ non-renormalisable: All amplitudes are divergent at a sufficiently high order.

other example $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{n!}\phi^n$

$$D = d - \underbrace{\left[d - \left(\frac{d-2}{2}\right)n\right]}_{\text{mass dim. of coupling}} V - \underbrace{\left(\frac{d-2}{2}\right)}_{\text{mass dim. of } \phi} N$$

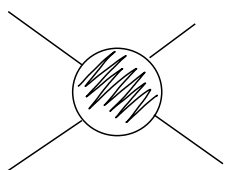
$d=4, n=4$ renormalisable

7.2. Renormalised ϕ^4 -theory

$D = 4 - N$ $\mathbb{Z}_2$ -sym $\phi \rightarrow -\phi \Rightarrow$ diagrams with odd N vanish

a)  $D=4$ b)  $D=2$

Vacuum shift $\sim \Lambda^2 + p^2 \log \Lambda + \text{finite}$

c)  $D=0$ $\sim \log \Lambda + \text{finite}$

} 3 $\propto$ constants absorb them into bare mass, coupling and field strength

① rescale the field strength

$$\phi = z^{1/2} \phi_r \quad \nearrow \text{see EX 8.i)}$$

② eliminate m_0, λ_0 from $\mathcal{L}$ by

$$\delta z = z - 1, \quad \delta m = m_0^2 z - m^2, \quad \delta \lambda = \lambda_0 z^2 - \lambda$$

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi_r)^2 - \frac{1}{2} m^2 \phi_r^2 - \frac{\lambda}{4!} \phi_r^4$$

$$\boxed{+ \frac{1}{2} \delta z (\partial_\mu \phi_r)^2 - \frac{1}{2} \delta m \phi_r^2 - \frac{\delta \lambda}{4!} \phi_r^4}$$

counterterms; absorb ∞ , but unobservable, shift between bare parameters and physical ones

Additional Feynman rules:

a) Propagator: $D_F(p) = \frac{i}{(1 + \delta z) p^2 - (1 + \delta m) m^2}$

$$D_F(p) \approx \frac{i}{p^2 - m^2} \left[1 + i (\delta z p^2 - \delta m) \frac{i}{p^2 - m^2} \right]$$

b)

$$\begin{array}{l} \text{---} \bigotimes \text{---} \xrightarrow{p} = i (p^2 \delta z - \delta m) \\ \text{---} \bigotimes \text{---} = -i \delta \lambda \end{array} \left. \vphantom{\begin{array}{l} \text{---} \bigotimes \text{---} \xrightarrow{p} \\ \text{---} \bigotimes \text{---} \end{array}} \right\} \begin{array}{l} \text{both with} \\ \text{amputated} \\ \text{legs} \end{array}$$

Prescription: Include these counter terms and adjust $\delta z, \delta m$ and $\delta \lambda$ such that

$$\text{---} \bigcirc \text{---} = \frac{i}{p^2 - m^2} + \text{terms reg at } p^2 = m^2 \quad \text{and}$$

$$\text{---} \bigcirc \text{---} = -i \lambda$$

at $s = 4m^2, t = u = 0$

hold. The conditions are called renormalisation conditions.

At one loop:

$$iM(p_1 p_2 \rightarrow p_3 p_4) = \text{diagram with a shaded circle and four external lines } p_1, p_2, p_3, p_4 = \text{diagram 1} + \text{diagram 2} + \dots$$

$$(\text{diagram 1} + \text{diagram 2} + \text{diagram 3}) + \dots$$

Mandelstam variables

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2$$

$$t = (p_1 - p_3)^2 = (p_4 - p_2)^2$$

$$u = (p_1 - p_4)^2 = (p_3 - p_2)^2$$

$$\text{diagram of a loop} = \frac{(-i\lambda)^2}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2} \frac{i}{(k+p)^2 - m^2}$$

$\uparrow p = p_1 + p_2$ $\circ = (-i\lambda)^2 \cdot iV(p^2)$ counter term

$$iM = -i\lambda + (-i\lambda)^2 [iV(s) + iV(t) + iV(u)] - i\delta\lambda$$

$$iM|_{s=4m^2, t=u=0} = -i\lambda \rightarrow \delta\lambda = -\lambda^2 [V(4m^2) + 2V(0)]$$

after dimensional regularisation with $\epsilon = 4 - d$

$$V(p^2) = -\frac{1}{32\pi^2} \int_0^1 dx \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log[m^2 - x(1-x)p^2] \right)$$

$$\delta\lambda = \frac{\lambda^2}{32\pi^2} \int_0^1 dx \left(\frac{6}{\epsilon} - 3\gamma + 3\log(4\pi) - \log[m^2 - x(1-x)4m^2] - 2\log m^2 \right)$$

$$iM = -i\lambda - \frac{i\lambda^2}{32\pi^2} \int_0^1 dx \left[\log\left(\frac{m^2 - x(1-x)s}{m^2 - x(1-x)4m^2}\right) + \log\left(\frac{m^2 - x(1-x)t}{m^2}\right) + \log\left(\frac{m^2 - x(1-x)u}{m^2}\right) \right]$$

divergence free?

remember: for the two point function we found:

$$\text{diagram of a loop} = \text{diagram 1} + \text{diagram 2} + \dots \quad \text{geometric series}$$

$$= \frac{i}{p^2 - m^2 - M^2(p^2)} \quad \text{have pole with residue 1 at } p^2 = m^2$$

$$\rightarrow M^2(p^2)|_{p^2=m^2} = 0 \text{ and } \frac{d}{dp^2} M^2(p^2)|_{p^2=m^2} = 0$$

at one-loop: $-iM^2(p^2) = \text{---} \bigcirc \text{---} + \text{---} \bigotimes \text{---}$

$$= \underbrace{-\frac{i\lambda}{2} \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(1-d/2)}{(m^2)^{1-d/2}}}_{= -i\lambda/2 \cdot \int \frac{d^d k}{(2\pi)^d} \cdot \frac{i}{k^2 - m^2}} + i(p^2 \delta_2 - \delta_m)$$

$\rightarrow \delta_2 = 0$ and $\delta_m = -\frac{\lambda}{2(4\pi)^{d/2}} \frac{\Gamma(1-d/2)}{(m^2)^{1-d/2}}$

$\nwarrow$ first contribution at two-loops from

$\text{---} \bigcirc \text{---} + \text{---} \bigcirc \bigotimes \text{---} + \text{---} \bigotimes \text{---}$

$\sim \lambda^2 \quad \quad \quad \sim \lambda^2$