An Introduction to String Theory, Winter 2022/23

Lecturer: Dr. Falk Hassler, falk.hassler@uwr.edu.pl
Assistant: M.Sc. Luca Scala, 339123@uwr.edu.pl

11. Effective action and compactification (21 points)

To be discussed on Thursday, 26th January, 2023 in the tutorial.
Please indicate your preferences until Saturday, 21/01/2023, 21:00:00 on the website.

Exercise 11.1: Effective action

Consider the effective action

$$S = \int \mathrm{d}^D x \sqrt{-G} e^{-2\phi} \left[R + 4(\nabla\phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right] \quad \text{with} \quad H_{\mu\nu\rho} = 3\partial_{[\mu} B_{\nu\rho]}$$

for the massless closed string excitations in the low energy limit.

a) (1 point) Explain how the fields $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ in this action are related to the massless modes of the string we found after quantising it in Minkowski space.

Compute the field equations for

b) (2 points) the metric $g_{\mu\nu}$,

c) (2 points) the B-field $B_{\mu\nu}$,

d) (1 point) the dilaton ϕ.

e) (2 points) How do you combine these field equations to obtain the one-loop β-functions

$$\begin{aligned} \beta^G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{4} H_\mu{}^{\rho\sigma} H_{\nu\rho\sigma} + 2\nabla_\mu \nabla_\nu \phi \\ \beta^B_{\mu\nu} &= -\frac{1}{2} \nabla_\lambda H^\lambda{}_{\mu\nu} + H^\lambda{}_{\mu\nu} \nabla_\lambda \phi \\ \beta^\phi &= (\nabla\phi)^2 - \frac{1}{2}\nabla^2\phi - \frac{1}{24} H_{\mu\nu\rho} H^{\mu\nu\rho} \,. \end{aligned}$$

of the critical string.

Exercise 11.2: Kaluza-Klein reduction

To compactify gravity from $D+1$ to D dimensions we make the ansatz

$$G_{MN} = \phi^\beta \begin{pmatrix} g_{\mu\nu} + \phi A_\mu A_\nu & \phi A_\mu \\ \phi A_\nu & \phi \end{pmatrix} \tag{1}$$

for the metric and consider the Einstein-Hilbert action in $D+1$ dimensions

$$S_{D+1} = \frac{1}{16\pi\kappa_{D+1}} \int \mathrm{d}^{D+1} x \sqrt{-G} R_{D+1} \,.$$

a) (1 point) Compute the inverse metric G^{MN}.

b) (4 points) Write the Ricci scalar R_{D+1} in $D+1$ dimensions in term of the one in D dimensions, R_D, the electro-magnetic field strength tensor $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ and ϕ.

Hint: Write down R_{D+1} in terms of the metric G_{MN} and its inverse. Use the decomposition $x^M = \begin{pmatrix} x^\mu & y \end{pmatrix}$ of the coordinates and keep in mind that nothing depends on the y coordinate ($\partial_y \cdots = 0$).

c) (2 points) Compute $\sqrt{-G}$ by using the decomposition (1).

Hint: It helps to decompose G_{MN} into the product of three matrices, $G = ABA^T$. The matrix A can be chosen to be triangular and B is block diagonal.

d) (2 points) Determine β as a function of the dimension D such that the reduced action becomes

$$S_{D+1} = \frac{1}{16\pi\kappa_D} \int \mathrm{d}^D x \sqrt{-g} R_D + \ldots \tag{2}$$

The choice of β for which no ϕ appears in front of the Ricci scalar is called Einstein frame.

Hint: You do not have to do the full reduction. Just use $\sqrt{-G}$ from the previous task and assume that ϕ=const. in R_{D+1}.

e) (2 points) What is the D dimensional Newton constant κ_D in terms of the radius R of the compact space[1]?

f) (2 points) Show that $D+1$ dimensional generalised coordinate transformations

$$G_{MN} \to \frac{\partial x^R}{\partial x'^M} \frac{\partial x^S}{\partial x'^N} G_{RS}$$

induce D dimensional gauge transformations when reparameterising the circle coordinate as $y \to y + \lambda(x^\mu)$and $x^\mu \to x^\mu$.

[1] Clearly with only one compact dimension this has to be a circle.