Quanten Field Theory Summer 2022

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3. The Eight Fold Way

To be discussed on Tuesday, 22nd March, 2022 in the seminar.

During the lecture we got puzzled about how to identify pions from the isospin triplet. In this exercise we want to look at this problem and, while doing so, learn more about the Lie algebra for the Lie group SU(3).

Exercise 3.1: A Primer in Lie Groups and Lie Algebras

Particularly important for our discussion is the Lie group SU(3) and its Lie algebra. It consists of all unitary 3×3 matrices U, whose determinant is one.

a) Like in the lecture for SU(2), we want to write them as

$$U(\alpha^J) = e^{i\alpha^J \lambda_J} \tag{1}$$

with $J = 1, \dots, 8$ and eight linearly independent, hermitian matrices λ_J.
Show that $\lambda_I^\dagger = \lambda_I$ implies that U is unitary ($U^\dagger = U^{-1}$). But, this is not enough, we require moreover that $\det U = 1$, too. Explain why this is is only possible if all λ_I's are traceless. Finally prove that all traceless hermitian 3×3 matrices λ can be written as the sum $\lambda = \alpha^J \lambda_J$ over eight different λ_J's (assuming the α^J's are real).

b) One can pick two generators out of the eight which commute. We pick

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad H_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} . \tag{2}$$

All six remaining generators can be written as eigenvectors under the adjoint action of H_i, $i = 1, 2$,

$$[H_i, E_{\vec\alpha}] = \alpha_i E_{\vec\alpha} . \tag{3}$$

Hence, in total you should find six different tuples of eigenvalues (α_1, α_2). Plot them and show that they are on a grid spanned by $(1/2, \sqrt{3}/2)$ and $(1/2, -\sqrt{3}/2)$, called root lattice of SU(3).

c) Show that the two roots on the $\alpha_2 = 0$ axis form an SU(2) subalgebra after an appropriate choice of a third generator. Finally, find the coefficients in $Q = c_1 H_1 + c_2 H_2$ such that the adjoint action $[Q, E_\alpha] = q_{\vec\alpha} E_{\vec\alpha}$ has only eigenvalues -1, 0, or 1.

Exercise 3.2: The Physics Behind The Math

Now, we want relate this mathematical framework with some physics!

a) Show that the four components of the Dirac spinor ψ capture a fermion with spin $\pm 1/2$ and that $\overline{\psi}$ describes the corresponding anti-particle (see problems 1.c) and 1.d) from the first exercise sheet). We can now identity the spinors U acts on as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \qquad \text{and} \qquad \overline{\psi} = \begin{pmatrix} \overline{\psi}_1 & \overline{\psi}_2 & \overline{\psi}_3 \end{pmatrix} = \begin{pmatrix} \overline{u} & \overline{d} & \overline{s} \end{pmatrix} . \tag{4}$$

What is the physical interpretation of u, d, s and their bared counterparts?

b) Show that the combination $\overline{\psi}^i \psi_j$, with i and $j = 1, 2, 3$ is invariant under Lorentz transformations. We can therefore identify it with pseudoscalar mesons. Take all these mesons and organise them into the plot of the root diagram according to their electric charge Q and strangeness $S = \alpha_2$.

Exercise 3.3: Taking Spin into Account

Waring: This is a hard problem. But it is fun, too :-). If you don't have a solid background in representation theory you might avoid it. Most likely I will present it and show you how to have some fun stuff with Mathematica if we have the time.

As we have seen in the last problem, there is not only the SU(3) flavor symmetry of the quarks (which we have deal with above) but also another SU(2) that originates from their spin. If we take both into account, quarks transform in the product irrep $\mathbf{3} \otimes \mathbf{2}$ of SU(3)×SU(2). Explore this idea to obtain not only the pseudo-scalar meson octet from above but the baryon octet and decuplet, too.