

11. BRST Symmetry

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remember: Quantisation of (non-)abelian gauge field required gauge fixing \Rightarrow we lose gauge symmetry
Today, we restore it with BRST.

11.1. The Faddeev-Popov Lagrangian

goal: evaluate path integral $I = \int DA \exp\left[-\frac{i}{4} \int d^4x (F_{\mu\nu}^a)^2\right]$



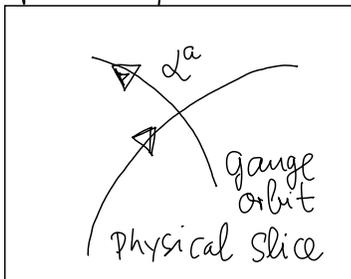
trick remove redundancy by gauge fixing cond.

$$G(A) = 0 \quad \text{and}$$

$$1 = \int D\alpha(x) \delta(G(A^\alpha)) \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right)$$

with $(A^\alpha)_\mu^a = e^{i\alpha^a t_a} \left[A_\mu^b t_b + \frac{i}{g} \partial_\mu \right] e^{-i\alpha^c t_c}$

field space



and infinitesimal version

$$\begin{aligned} (A^\alpha)_\mu^a &= A_\mu^a + \frac{1}{g} \partial_\mu \alpha^a + f^{bc a} A_\mu^b \alpha^c \\ &= A_\mu^a + \frac{1}{g} D_\mu \alpha^a \quad (1) \end{aligned}$$

acting on α^a in adjoint representation

now we can write:

$$I = \left(\int D\alpha \right) \int DA e^{iS[A]} \delta(G(A)) \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right)$$

in generalised Lorenz gauge, $G(A) = \partial^\mu A_\mu^a(x) - \omega^a(x)$, (2)

we eventually find:

$$\langle A_\mu^a(x) A_\nu^b(y) \rangle = \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 + i\epsilon} \left(g_{\mu\nu} - (1-\xi) \frac{k_\mu k_\nu}{k^2} \right) \delta^{ab} e^{-ik(x-y)}$$

remember: $\xi=1$ is called Feynman-'t Hooft gauge

⚡ Do not forget the $\det(\dots)$! Because we have

$$\frac{\delta G(A^a)}{\delta \alpha} = \frac{1}{g} \partial^\mu D_\mu, \text{ instead of just a constant factor (in abelian version)}$$

→ results in ghost field Lagrangian

$$\det\left(\frac{1}{g} \partial^\mu D_\mu\right) = \int \mathcal{D}c \mathcal{D}\bar{c} \exp\left[i \int d^4x \underbrace{\bar{c}(-\partial^\mu D_\mu)c}_{\text{ghost}}\right]$$

ghost $\hat{=}$ violates spin-statistic, i.e. $\mathcal{L}_{\text{ghost}}$

Spin 0 but fermionic path integral $\Rightarrow c = \text{Grassmann var.}$

$$\mathcal{L}_{\text{ghost}} = \bar{c}^a \left(-\partial^2 \delta_a^c - g f_{ab}^c \partial_\mu A^{b\mu} \right) c_c$$

with propagator: $\langle c_a(x) \bar{c}^b(y) \rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2} \delta_a^b e^{-ik(x-y)}$

and Feynman rules:

$$a \text{ --- } b = \frac{i \delta_a^b}{p^2},$$

$$= -g f_{ab}^c p_\mu$$

11.2. The BRST Lagrangian

Question: Can we still find the original gauge transformations in the gauge fixed action?

Idea: Introduce a new (commuting) scalar field B_a

$$\mathcal{L} = -\frac{1}{4} (F_{\mu\nu}^a)^2 + \bar{\psi} (i\not{D} - m) \psi - \frac{\xi}{2} (B_a)^2 + B_a \partial^\mu A_\mu^a + \bar{c}^a (-\partial^\mu D_\mu)_a^b c_b$$

B_a has no kinetic term \rightarrow auxiliary field
we can integrate it out

This Lagrangian has the global symmetry:

$$\delta A_\mu^a = \varepsilon (D_\mu)^{ab} C_b$$

$$\delta \psi = i g \varepsilon C_a t^a \psi$$

$$\delta c_a = -\frac{1}{2} g \varepsilon f^{bc}{}_a C_b C_c$$

$$\delta \bar{c}^a = \varepsilon B^a$$

$$\delta B^a = 0$$

↑ local gauge

transformation with $\rightarrow d^a(x) = g \varepsilon C^a(x)$
 comm. anti-comm.

We denote this transformation with

$$\delta \phi := \varepsilon Q \phi \leftarrow \text{any field, i.e. } Q A_\mu^a = (D_\mu)^{ab} C_b$$

One can now check:

$$\boxed{Q^2 \phi = 0}, \text{ i.e. } Q^2 c_a = \frac{1}{2} g^2 f^{be}{}_a f^{cd}{}_e C_b C_c C_d$$

$$\text{Jacobi identity} \rightarrow = 0$$

In the Hamiltonian picture Q commutes with the Hamilton operator H , $[Q, H] = 0$, $Q^2 = 0$

→ eigenstates of H decompose into:

1) $|\psi_1\rangle$ with $Q|\psi_1\rangle \neq 0$

2) $|\psi_2\rangle$ with $|\psi_2\rangle = Q|\psi_1\rangle$ for a particular $|\psi_1\rangle$

3) $|\psi_0\rangle$ with $Q|\psi_0\rangle = 0$ and
 $|\psi_0\rangle \neq Q|\psi_1\rangle$ for any $|\psi_1\rangle$

and the Hilbert space decomposes into $\mathcal{H}_i \in |\psi_i\rangle$

For single particle states: \mathcal{H}_1 has forward gauge bosons and anti-ghosts

physical fields → \mathcal{H}_2 has backward gauge bosons and ghosts

$$\boxed{\mathcal{H}_0 \text{ has transverse gauge bosons}}$$

compare with exterior derivative: $d\phi = \partial_\mu \phi dx^\mu$ with
 $d^2 = 0$ and $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge d\omega_2$

$$\left. \begin{array}{l} d\omega = 0 \quad \hat{=} \quad \text{closed } n\text{-form} \\ \omega = d\lambda \quad \hat{=} \quad \text{exact } n\text{-form} \end{array} \right\} n = \deg \omega$$

de Rham cohomology: $H_{dR}^k(M) = \frac{\text{closed}}{\text{exact}} k\text{-forms}$
manifold

analogy: $d \hat{=} Q$

physical states are defined by BRST cohomology