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2. Relativistic Actions (17 points)

To be discussed on Thursday, 13th October, 2022 in the tutorial.
Please indicate your preferences until Saturday, 08/10/2022, 21:00:00 on the website.

Exercise 2.1: Point particle action

Consider the action for a point particle with the dynamical einbein $e(\tau)$ and $X(\tau)$ fields:

$$S = \frac{1}{2} \int \mathrm{d}\tau \, \left(e^{-1} \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} - e m^2 \right) . \tag{1}$$

a) (1 point) Derive the equations of motion.

b) (1 point) Integrate out the field e to discover the first relativistic point particle action

$$S_{\mathrm{pp}} = -m \int \mathrm{d}\tau \, \sqrt{-\dot{X}^\mu \dot{X}_\mu} \tag{2}$$

discussed in the lecture.

Exercise 2.2: Non-relativistic limit of the point particle action 2 points

Expand the action (2) in the non-relativistic limit and show that it becomes the action for a point particle of mass m.

Exercise 2.3: Reparametrisation invariance

The action (1) is invariant under reparametrisation of the particle world line.

a) (2 points) Consider the finite transformation

$$\tau \to \tau' = f(\tau)$$

and show the reparametrisation invariance of the action.

b) (2 points) Now do the same with the infinitesimal version,

$$\tau \to \tau' = \tau - \xi(\tau) \, ,$$

without using the results from the previous task.

Exercise 2.4: Point particle in curved space 3 points

Hint: You might want to only choose this problem when you have already some familiarity with general relativity.

Take the action (1) with a general metric $g_{\mu\nu}$ instead of the Minkowski metric $\eta_{\mu\nu}$,

$$S = \frac{1}{2} \int \mathrm{d}\tau \, \left(e^{-1} \dot{X}^\mu \dot{X}^\nu g_{\mu\nu} - e m^2 \right) \tag{4}$$

and compute its equation of motion in a gauge with $e = 1$. Show that its the same as the geodesic equation.

Exercise 2.5: *p*-brane action 3 points

We can easily generalise the Nambu-Goto action from the lecture to p-branes (where the string has $p = 1$):

$$S_p = -T_p \int \mathrm{d}^{p+1}\sigma \sqrt{-\det G_{\alpha\beta}}$$

with the induced metric

$$G_{\alpha\beta} = g_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu \, . \tag{5}$$

Show that this action is invariant under reparametrisations.

Exercise 2.6: Induced metric for the two-sphere 3 points

Compute the metric of the two-sphere with radius R by embedding it into Euclidean space and using the (5).