An Introduction to String Theory, Winter 2022/23

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8. Path integrals, ghosts and Grassmann numbers (19 points)

To be discussed on Thursday, 1st December, 2022 in the tutorial.
Please indicate your preferences until Saturday, 26/11/2022, 21:00:00 on the website.

Exercise 8.1: Gaussian integrals

Your background on path integrals might vary. Therefore, in this exercise we try to bring everyone up to speed and compute Gaussian path integrals from scratch. The most important trick is to start with integral over a finite number n of variables. In the following, integration over $\mathbb{R}^n$ is always understood.

a) (1 point) Compute the one-dimensional Gauss integral

$$I = \int_{-\infty}^{\infty} \mathrm{d}x \, e^{-\frac{a}{2}x^2} \,.$$

b) (2 points) Extend this computation to $\mathbb{R}^n$ to evaluate

$$Z(0) := \int d^n\vec{x} \, e^{-\frac{1}{2}\vec{x}^T A \vec{x}} \,, \quad A^T = A \,.$$

Hint: Apply a similarity transformation to A to make it diagonal.

c) (2 points) To compute correlation functions, we have to add a source current to the partition functions, resulting in

$$Z(\vec{j}) := \int d^n x \, e^{-\frac{1}{2}\vec{x}^T A \vec{x} + \vec{j}^T \vec{x}} \,, \quad \vec{j} \in \mathbb{R}^n \,. \tag{1}$$

Compute $Z(\vec{j})$ explicitly.
Hint: The trick here is to complete the square.

Next, we turn to the expectation value of a function $B(\vec{x})$, $\langle B(\vec{x}) \rangle$. It can be computed by

$$\langle B(\vec{x}) \rangle := \frac{1}{Z(0)} \int d^n\vec{x} \, B(\vec{x}) e^{-\frac{1}{2}\vec{x}^T A \vec{x}} \,.$$

d) (1 point) Show that $\langle 1 \rangle = 1$.

e) (1 point) Why is

$$\int d^n\vec{x} \frac{\partial}{\partial x^i} \left(B(\vec{x}) e^{-\frac{1}{2}\vec{x}^T A \vec{x}} \right) = 0 \,?$$

f) (2 points) The true power of the correlation function is that it easily admits us to compute higher correlation functions (those that depend on more than one random vector $\vec{x}$) by

$$\langle B(\vec{x}_1, \dots, \vec{x}_n) \rangle = \left. \frac{B\left(\frac{\partial}{\partial j_1}, \dots, \frac{\partial}{\partial j_n}\right) Z(\vec{j})}{Z(0)} \right|_{\vec{j}=0} \tag{2}$$

Prove this relation.

g) (1 point) Compute $\langle x_i x_j \rangle$ by using (2)

h) (2 points) and $\langle x_i x_j x_k x_l \rangle$ in the same way.

i) (1 bonus point) Reproduce the result from the last task by applying Wick's theorem.

Exercise 8.2: Faddeev-Popov ghosts and Grassmann numbers

Determinants of operators such as the Faddeev-Popov determinant $\Delta_{\text{FP}} = \det P$ from the lecture can formally be written as a separate path integral over a new set of auxiliary variables. In order for this to be possible, these auxiliary variables have to be anti-commuting rather than ordinary commuting numbers. Two anti-commuting numbers (or Grassmann numbers) ϕ and η satisfy

$$\phi\eta = -\eta\phi$$

and hence $\phi^2 = 0$. Because of this, the most general function of a Grassmann variable ϕ is

$$f(\phi) = A + B\phi$$

with A, $B \in \mathbb{C}$.

Integrals over Grassmann variables ("Berezin integrals") are defined by

$$\int d\phi [A + B\phi] := B\,. \tag{3}$$

These properties mimic similar properties of ordinary integrals of the type $\int_{-\infty}^{\infty} dx f(x)$, which is the motivation for the unusual definition (3). Note that, for Grassmann variables, integration and differentiation are equivalent operations.

If one has several linearly independent Grassmann variables ϕ_i $(i = 1, \ldots, n)$, where

$$\phi_i \phi_j = -\phi_j \phi_i \quad \forall\, i\,, j\,,$$

one defines

$$\int d\phi_1 \ldots d\phi_n f(\phi_i) = c\,,$$

where c is the coefficient in front of the $\phi_n \phi_{n-1} \ldots \phi_1$-term in $f(\phi^i)$ (note the order):

$$f = \cdots + c\phi_n \phi_{n-1} \ldots \phi_1\,.$$

a) (2 points) Let n be even and split the ϕ_i into two sets ψ_m, χ_m $(m = 1, \ldots, \frac{n}{2})$:

$$(\phi_1, \ldots \phi_n) = (\psi_1, \chi_1, \psi_2, \chi_2, \ldots, \psi_{\frac{n}{2}}, \chi_{\frac{n}{2}})\,.$$

Show that

$$\Big(\prod_{m=1}^{\frac{n}{2}} \int d\psi_m d\xi_m\Big) e^{\sum\limits_{m=1}^{\frac{n}{2}} \chi_m \lambda_m \psi_m} = \prod_{m=1}^{\frac{n}{2}} \lambda_m\,,$$

where $\lambda_m \in \mathbb{C}$ are ordinary c-numbers[1] and the exponential is defined via its power series expansion.

[1]c-numbers denote ordinary commuting numbers.

If the λ_m are the eigenvalues of an operator Λ, one thus obtains

$$\Big(\prod_{m=1}^{\frac{n}{2}} \int d\psi_m d\chi_m\Big) e^{\sum\limits_{m,l=1}^{\frac{n}{2}} \chi_m \Lambda_{ml} \psi_l} = \det \Lambda \,,$$

or, in a path integral context with Grassmann-valued fields $\psi(x)$, $\chi(x)$ and a differential operator that we call here Δ,

$$\int \mathcal{D}[\psi]\mathcal{D}[\chi] e^{\int d^d x \chi \Delta \psi} = \det \Delta \,.$$

Using similar arguments (see, e.g. Polchinski, Chapter 3.3 for a detailed account), one obtains

$$\det P = \int \mathcal{D}[c_\alpha]\mathcal{D}[b^{\beta\gamma}] \exp\left[-\frac{i}{4\pi}\int d^2\sigma \sqrt{h} b^{\alpha\beta}(Pc)_{\alpha\beta}\right] ,$$

where $b^{\alpha\beta}(\sigma) = \beta^{\beta\alpha}(\sigma)$ is a symmetric traceless anti-commuting field, and $c_\alpha(\sigma)$ is an anti-commuting world sheet vector field.

b) (2 points) Show that, due to the symmetry and tracelessness of $b^{\alpha\beta}$, one can write

$$\det P = \int \mathcal{D}[c_\alpha]\mathcal{D}[b^{\beta\gamma}] \exp\left[-\frac{i}{2\pi}\int d^2\sigma \sqrt{h} b^{\alpha\beta} \nabla_\alpha c_\beta\right]$$

by using the P we discussed in the lecture.

It is more convenient to use $b_{\alpha\beta}$ ("anti-ghost") and c^α ("ghost") as the independent fields, as they turn out to be neutral under Weyl transformations, whereas $b^{\alpha\beta}$ and c_α are not due to additional powers of the (inverse) metric.

c) (1 point) Use therefore

$$S_{\text{ghost}} = -\frac{i}{2\pi}\int d^2\sigma \sqrt{h} b_{\alpha\beta} \nabla^\alpha c^\beta$$

to derive the ghost action in flat worldsheet light cone coordinates,

$$S_{\text{ghost}} = \frac{i}{\pi}\int d^2\sigma \left(c^+ \partial_- b_{++} + c^- \partial_+ b_{--}\right) . \tag{4}$$

d) (1 point) Derive the equations of motion for $c^\pm$ and $b_{\pm\pm}$ from (4).

The total gauge fixed path integral is now

$$Z = \int \mathcal{D}[X]\mathcal{D}[c]\mathcal{D}[b] e^{i[S_{\text{P}} + S_{\text{ghost}}]|_{h_{\alpha\beta}=\eta_{\alpha\beta}}} ,$$

and one clearly sees that it would have been inconsistent to simply set $h_{\alpha\beta} = \eta_{\alpha\beta}$ and drop the $\mathcal{D}[h]$ integration, as that would have missed the ghost contribution. To appreciate the ghost contribution, one notes that the total energy momentum tensor $T_{\alpha\beta}$ now also gets a contribution from the ghost action

$$T_{\alpha\beta} = T_{\alpha\beta}[X] + T_{\alpha\beta}[b,c]$$

which modifies the central charge term in the Virasoro algebra to

$$A(m) = \frac{D}{12} m(m^2-1) + \frac{1}{6}(m - 13m^3) - 2am \,.$$

A non-vanishing total $A(m)$ translates to an anomaly of the local Weyl transformations.

e) (1 point) Verify that this anomaly is absent if and only if $D = 26$ and $a = -1$.