

6. Loop Effects

Motivation: We have the path integral & Feynman diagrams.
Time to do some physics!

Remember EX5: physical mass m_{phys} ($\neq m$ in Lagrangian)
from 1-particle irreducible (1PI) function.

Today more details.

6.1. Field-Strength Renormalisation

Take Hamiltonian of interacting scalar theory which is invariant under space and time translations.

Conserved charges: $P_\mu \xrightarrow{\text{quantisation}} \text{operator } \hat{P}_\mu$

$[\hat{P}_\mu, \hat{H}] = 0 \rightarrow$ have same eigenstates $P_\mu |\lambda_p\rangle = \hat{P}_\mu |\lambda_p\rangle$

Completeness: $1 = |0\rangle\langle 0| + \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p(\lambda)} |\lambda_p\rangle\langle \lambda_p|$

$E_p(\lambda) = \sqrt{|\vec{p}|^2 + m_\lambda^2}$ energy of state $|\lambda_p\rangle$ with mass m_λ

For $x^0 > y^0$:

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p(\lambda)} \langle 0 | \phi(x) | \lambda_p \rangle \langle \lambda_p | \phi(y) | 0 \rangle$$

$$\langle 0 | \phi(x) | \lambda_p \rangle = \langle 0 | e^{i\hat{P}x} \phi(0) e^{-i\hat{P}x} | \lambda_p \rangle$$

$$= \langle 0 | \phi(0) | \lambda_p \rangle e^{-iPx} \quad |p^0 = E_p$$

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$$\phi(0) = U_p \phi(0) U_p^{-1} \text{ \& }$$

$$\langle 0 | U_p = \langle 0 |$$

$$|\lambda_p\rangle = U_p |\lambda_0\rangle$$

Lorentz boost $\rightarrow$

(vacuum is Lorentz invariant)

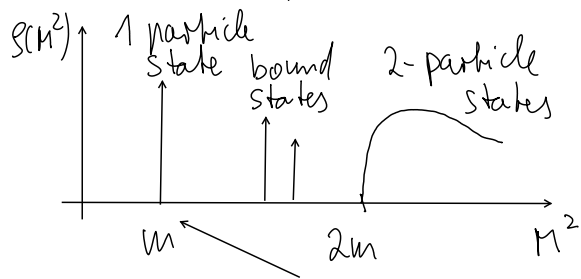
$$= \sum_\lambda \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m_\lambda^2 + i\epsilon} e^{-ip(x-y)} |\langle 0 | \phi(0) | \lambda_0 \rangle|^2$$

Same for $y^0 > x^0 \leadsto \boxed{\langle 0 | T \phi(x) \phi(y) | 0 \rangle = \int \frac{dM^2}{2\pi} \rho(M^2) D_F(x-y, M^2)}$

Källén - Lehmann spectral representation

with positive spectral density function

$$\rho(M^2) = \sum_{\lambda} (2\pi) \delta(M^2 - m_{\lambda}^2) |\langle 0 | \phi(0) | \lambda_0 \rangle|^2$$



All this just from symmetry!

Similar for higher spins.

$$\rho(M^2) = 2\pi \delta(M^2 - m^2) \cdot Z + \text{rest}$$

field strength renormalisation

Compare to EX 5: $m = m_{\text{phys}}$ and the mass parameter in Lagrangian is $m_0 = \text{bare mass}$

6.2. Electron Self-Energy

So far only symmetries, now Feynman diagrams in QED

$$\langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle = X \leftarrow Y + X \text{ (loop) } \leftarrow Y + \dots$$

with the free field propagator $\leftarrow_P = \frac{i(\not{P} + m_0)}{P^2 - m_0^2 + i\epsilon}$
the second diagram evaluates to

$$\text{Diagram: } \leftarrow_P \text{ (with a loop) } \leftarrow_P = \frac{i(\not{P} + m_0)}{P^2 - m_0^2} \left[-i \Sigma_2(P) \right] \frac{i(\not{P} + m_0)}{P^2 - m_0^2} \quad \text{with}$$

$$-i \Sigma_2(P) = (-ie)^2 \int \frac{d^4 k}{(2\pi)^4} \gamma^\mu \frac{i(\not{k} + m_0)}{k^2 - m_0^2 + i\epsilon} \gamma_\mu \frac{-i}{(P-k)^2 - (\mu^2 + i\epsilon)}$$

(Remember $\mu \text{ wavy } \nu = \frac{-i g_{\mu\nu}}{q^2 + i\epsilon}$ and $\text{fermion vertex} = -ie \gamma^\mu$)

$\mu = \text{small mass for the photon to remove the divergence at } (P-k)^2 = 0 \text{ (IR divergence).}$

Trick: Feynman parameter $\frac{1}{A \cdot B} = \int_0^1 \frac{dx}{(xA + (1-x)B)^2}$

$$-i \Sigma_2(P) = -e^2 \int_0^1 dx \int \frac{d^4 \ell}{(2\pi)^4} \frac{-2x \not{P} + 4m_0}{(\ell^2 - \Delta + i\epsilon)^2}$$

$$l = k - xp \quad \text{and} \quad \Delta = -x(1-x)p^2 + x\mu^2 + (1-x)m_0^2$$

This integral is still divergent! We have to regularise it (details next lecture). Here Pauli-Villars

reg.:

$$\int \frac{d^4 l}{2\pi} \frac{1}{(l^2 - \Delta)^2} \rightarrow \frac{i}{(4\pi)^2} \int_0^\infty d\ell_E^2 \left(\frac{\ell_E^2}{(\ell_E^2 + \Delta)^2} - \frac{\ell_E^2}{(\ell_E^2 + \Delta_\Lambda)^2} \right)$$

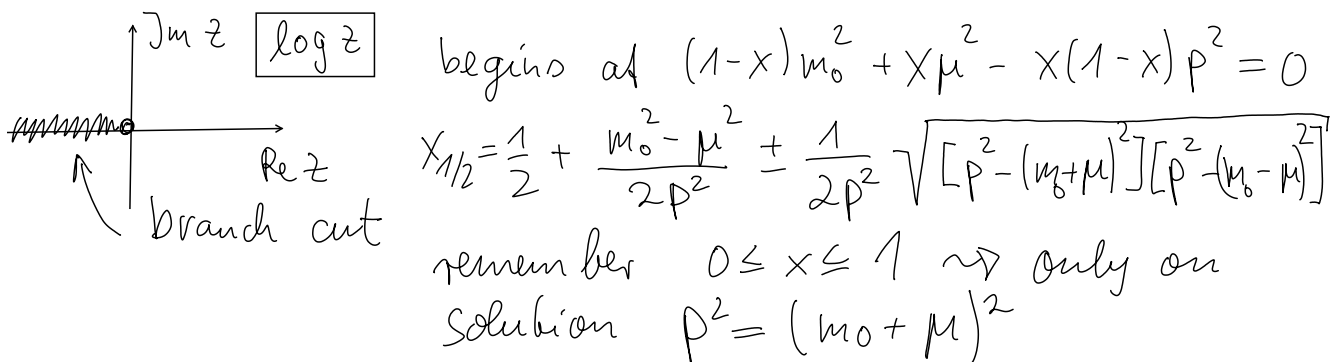
$$= \frac{i}{4\pi} \log\left(\frac{\Delta_\Lambda}{\Delta}\right)$$

- Λ = mass of photon, like μ , but we send $\Lambda \rightarrow \infty$
 $\rightarrow \Delta_\Lambda = -x(1-x)p^2 + x\Lambda^2 + (1-x)m_0^2 \xrightarrow{\Lambda \rightarrow \infty} x\Lambda^2$

- ℓ_E = Wick rotated 4-momentum $\ell_E^0 = -i\ell^0$

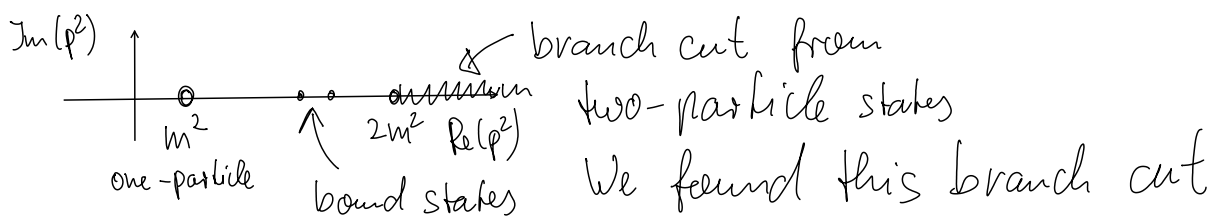
results in $\Sigma_2(p) = \frac{\alpha}{2\pi} \int_0^1 dx (2m_0 - xp) \log\left(\frac{x\Lambda^2}{(1-x)m_0^2 + x\mu^2 - x(1-x)p^2}\right)$

$$\alpha = \frac{e^2}{4\pi} = \text{fine structure constant}$$



This is the threshold to create two particles (electron + photon).

Compare with FT of $\langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle$:



Question: What about pole at m^2 ?

Answer: We need to sum an infinite series of diagrams.