An Introduction to String Theory, Winter 2022/23

Lecturer: Dr. Falk Hassler, falk.hassler@uwr.edu.pl
Assistant: M.Sc. Luca Scala, 339123@uwr.edu.pl

6. Virasoro Algebra (20 points)

To be discussed on Thursday, 17th November, 2022 in the tutorial.
Please indicate your preferences until Saturday, 12/11/2022, 21:00:00 on the website.

Exercise 6.1: Normal ordering and the quantum Virasoro algebra

In this exercise, we will show that in the quantised bosonic string theory, the normal ordered Virasoro generators

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} :\alpha_{m-n}\alpha_n:$$

satisfy the Virasoro algebra with a central charge[1]:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12} m(m^2-1)\delta_{m+n} \,.$$

In order to become more familiar with the normal ordering prescription, we will do this by brute force methods, i.e., by simply using the definition of the normal ordered generators L_m and then calculating their commutators. We will proceed in several smaller steps. At the end, we will use Wick's theorem, as already discussed in the lecture, and see how it makes the computation much easier.

a) (2 points) Explain why the normal ordering in L_m only affects L_0 and why the Virasoro generators L_m can be written in the following form:

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{0} \alpha_n \alpha_{m-n} + \frac{1}{2} \sum_{n=1}^{\infty} \alpha_{m-n}\alpha_n \,. \tag{1}$$

b) (2 points) Using $[X, Y, Z] = [X, Y]Z + Y[X, Z]$ and $[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n}\eta^{\mu\nu}$ prove that, for all $m, n \in \mathbb{Z}$

$$[\alpha^\mu_m, L_n] = m\alpha^\mu_{m+n} \,.$$

This is the pedestrian alternative to Wick's theorem, we used in the lecture.

c) (2 points) Decompose the sum

$$\sum_{n=-\infty}^{\infty} = \sum_{n=-\infty}^{0} + \sum_{n=1}^{\infty}$$

[1] A central charge, T_0, of a Lie algebra is a generator that commutes with all generators of the Lie algebra, $[T_a, T_0] = 0$, but appears on the right hand side of some commutators, $[T_a, T_b] = cT_0 + \dots$, for some T_a and T_b with c being a constant. In the above Virasoro algebra, the role of T_0 is played by the term proportional to δ_{m+n}, which should be viewed as an additional generator in addition to the L_m.

as we did in (1) to "solve" the normal ordering condition. Use the result of part b) to show that

$$
\begin{aligned}
[L_m, L_n] &= \frac{1}{2}\sum_{l=-\infty}^{0}\big((m-l)\alpha_l\alpha_{m+n-l} + l\alpha_{n+l}\alpha_{m-l}\big) \\
&+ \frac{1}{2}\sum_{l=1}^{\infty}\big((m-l)\alpha_{m+n-l}\alpha_l + l\alpha_{m-l}\alpha_{n+l}\big)\,.
\end{aligned}
\tag{2}
$$

d) (2 points) Make the substitution $p = n + l$ in the second and forth term in (2) and verify

$$
\begin{aligned}
[L_m, L_n] = \frac{1}{2}\Big(&\sum_{l=-\infty}^{0}(m-l)\alpha_l\alpha_{m+n-l} + \sum_{p=-\infty}^{n}(p-n)\alpha_p\alpha_{m+n-p} \\
&+ \sum_{l=1}^{\infty}(m-l)\alpha_{m+n-l}\alpha_l + \sum_{p=n+1}^{\infty}(p-n)\alpha_{m+n-p}\alpha_p\Big)\,.
\end{aligned}
\tag{3}
$$

e) (2 points) From now on, we will restrict ourselves to the case $n > 0$, as the other case $n < 0$ and $n = 0$ are completely analogous. Show therefore that for $n > 0$, the expression (3) is equal to

$$
\begin{aligned}
[L_m, L_n] = \frac{1}{2}\Big(&\sum_{p=-\infty}^{0}(m-n)\alpha_p\alpha_{m+n-p} + \sum_{p=1}^{n}(p-n)\alpha_p\alpha_{m+n-p} \\
&+ \sum_{p=n+1}^{\infty}(m-n)\alpha_{m+n-p}\alpha_p + \sum_{p=1}^{n}(m-p)\alpha_{m+n-p}\alpha_p\Big)\,.
\end{aligned}
\tag{4}
$$

Which of these four terms are already normal-ordered?

f) (2 points) Prove

$$
\sum_{p=1}^{n}(p-n)\alpha_p\alpha_{m+n-p} = \sum_{p=1}^{n}(p-n)\alpha_{m+n-p}\alpha_p + \sum_{p=1}^{n}(p-n)pD\delta_{m+n}
$$

and insert this for the second term in (4).

g) (2 points) Show that your result from part e) is now equivalent to

$$
[L_m, L_n] = \frac{1}{2}\sum_{l=-\infty}^{\infty}(m-n) :\alpha_l\alpha_{m+n-l}: + \frac{1}{2}D\sum_{l=1}^{n}(l^2 - nl)\delta_{m+n}\,. \tag{5}
$$

Prove, e.g. by induction, the following identities:

$$
\sum_{q=1}^{n} q^2 = \frac{1}{6}n(n+1)(2n+1)
$$

$$
\sum_{q=1}^{n} q = \frac{1}{2}n(n+1)
$$

and use them to finally derive

$$
[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}m(m^2-1)\delta_{m+n}\,.
$$

h) (2 points) Use Wick's theorem to compute

$$:a_1 a_2::a_3 a_4:=:a_1 a_2 a_3 a_4:+\dots\,, \tag{6}$$

where all operators on the right hand side are normal ordered.

i) (4 points) Use (6) to verify (5) from task g).

j) (4 bonus points) Try to implement what you did in task h) in Mathematica, or any other computer algebra system you think is capable to attack this problem.