4. Cartan - Weyl basis

4.1. Motivation

last lecture many new Lie groups : $U(N)$, $O(N)$, $Sp(2N)$, ...

Questions : • Are there more ?

• How do we find representations ?

Problem : Dealing with Lie group is complicated !

i.e. $SL(N)$ with $A \in M_{N \times N}$, $\det A = 1$

degree N polynomial in the matrix elements

→ find zeros of $\det A - 1 = 0$

only possible for $N \leq 4$ analytically

Solution to this problem is to transition to the Lie algebra

① consider adjoint action on group elements X

$$X' = A \cdot X \cdot A^{-1}$$

② make this action infinitesimal around the identity element $\mathbb{1}$

$$A = \mathbb{1} + \delta A \quad \rightarrow \quad A^{-1} = \mathbb{1} - \delta A$$

check : $AA^{-1} = A^{-1}A = \mathbb{1} - (\delta A)^2$ ← ignored because quadratic

③ take the same expansion for X to find :

$$\mathbb{1} + \delta X' = (\mathbb{1} + \delta A)(\mathbb{1} + \delta X)(\mathbb{1} - \delta A)$$

$$= \mathbb{1} + [\delta A, \delta X] + \dots \leftarrow \text{quadratic \& cubic terms}$$

$$\delta X' = [\delta A, \delta X]$$

adjoint representation of the corresponding Lie algebra

We haved used : 2) associativity

3) identity element

4) inverse element from the group axioms.

1) closure is encountered first at quadratic order where we require the Jacobi identity

$$[A, [B, C]] = [[A, B], C] + [B, [A, C]]$$

Result: Exponential map $\boxed{\exp: g \to G}$ relates elements of the Lie algebra g to elements of the Lie group G

$\delta A \in g$ $\qquad\qquad$ $A \in G$

$$\exp(\delta A) = \sum_{k=0}^{\infty} \frac{\delta A^k}{k!} = \mathbb{1} + \delta A + \frac{1}{2}(\delta A)^2 + \dots$$

⚡ There are some information about A lost in δA. I.e. the Lie groups $SU(2)$, $SO(3)$ and $O(3)$ share the same Lie algebra but they are not equivalent.

→ From now on we consider Lie algebras.

4.2. Adjoint representation

Def.: Let G be a Lie group and $g, h \in G$, then the adjoint action is defined as:

$$\boxed{Ad_g(h) := g \cdot h \cdot g^{-1}}$$

It is a group homomorphism $g \longmapsto Ad_g$:

$$Ad_{g_2} \circ Ad_{g_1}(h) = Ad_{g_2}(g_1 h g_1^{-1}) = g_2 g_1 h g_1^{-1} g_2^{-1}$$

composition becomes the group multiplication

$$= Ad_{g_2 \cdot g_1}$$

on the level of the Lie algebra, we have already seen

$$ad_X(Y) := [X, Y] \qquad X, Y \in \text{Lie algebra}$$

and again check:

$$[ad_{X_2}, ad_{X_1}](Y) = (ad_{X_2} \circ ad_{X_1} - ad_{X_1} \circ ad_{X_2})(Y) =$$

$$ad_{X_2}([X_1, Y]) - ad_{X_1}([X_2, Y]) = [X_2, [X_1, Y]] - [X_1, [X_2, Y]]$$

$$= [[X_2, X_1], Y] = ad_{[X_2, X_1]}(Y)$$

Jacobi identity

All informations we need to fix $ad_X(Y)$ are contained in the Lie algebra's structure coefficients

$$ad_{T_a}(T_b) = [T_a, T_b] = \sum_c f_{ab}{}^c T_c$$

see section 2.5

$ad\,T_a(\circ)$ is a representation, we find for any Lie algebra g

$$ad_{T_a}(T_b) = \sum_c (f_a)_b{}^c T_c$$

$1, \dots, \dim G$

$(f_a)_b{}^c$ are $\dim G$ different $(\dim G) \times (\dim G)$ matrices

4.3. Killing form

There is a natural inner product on the Lie algebra called the Killing form (or metric):

$$K(X,Y) := \frac{1}{I}\, tr(ad_X \circ ad_Y)$$

normalisation constant

trace in the adjoint matrix representation

In particular, for a matrix $g^a{}_b$, $tr\, g = \sum_a g^a{}_a$.

$$ad_{T_a} \circ ad_{T_b}(T_c) = ad_{T_a}([T_b, T_c]) = \sum_d f_{bc}{}^d\, ad_{T_a}(T_d)$$

$$= \sum_{d,e} \underbrace{f_{bc}{}^d f_{ad}{}^e}_{(g_{ab})_c{}^e} T_e$$

$$\rightarrow \boxed{K(T_a, T_b) = \kappa_{ab} = \frac{1}{I} \sum_{c,d} f_{ad}{}^c f_{bc}{}^d}$$

We can check that

$$K(X,Y) = K(Y,X) \qquad \text{symmetric}$$

$$K([X,Y],Z) + K(Y,[X,Z]) = 0 \qquad \text{invariant under adjoint action}$$

hold.

4.4. Cartan subalgebra

Def.: Let g be a Lie algebra, and $h \subset g$ a subalgebra.
h is called an ideal when $\forall x \in g, y \in h$, $[x,y] \in h$.

subalgebra: $\left.\begin{array}{l}[h,h] \subset h \\ [g,h] \subset h\end{array}\right\}$ $h =$ ideal

(I.) An algebra is called <u>simple</u> if it does not have any proper ideals.

strictly smaller than the whole algebra

(II.) An algebra is <u>semi-simple</u> if it is the direct sum of simple algebras.

Example: $U \in U(N) \rightarrow U^{+} = U^{-1}$

$U = \exp(i\,\delta u) \rightarrow \delta u^{+} = \delta u$

$\mathbb{1}^{+} = \mathbb{1}$ $\quad \delta u \rightarrow \mathbb{1}$ spans the subalgebra h

$\rightarrow$ all other N^2-1 traceless, hermitian matrices

$$\left.\begin{array}{ll}[\mathbb{1},\mathbb{1}] = 0\cdot\mathbb{1} & \text{subalgebra } \checkmark \\ [\mathbb{1},X] = 0\cdot\mathbb{1} & \checkmark\end{array}\right\} \text{ideal}$$

$\rightarrow$ $\mathbb{1}$ is a proper ideal in the Lie algebra $u(N)$

$\rightarrow$ $u(N)$ is not simple

by removing this ideal we get $SU(N)$ which is simple.
($U(1)$ - factor)

Cartan: An algebra is semi-simple iff (if and only if) the Killing form is non-singular.

To work with a semi-simple Lie algebra, it is convinient to use an eigenbasis $\{\tilde{T}_a\}$ with:

$$\mathrm{ad}_X(\tilde{T}_a) = [X,\tilde{T}_a] = \rho^a\,\tilde{T}_a$$

eigenvalues for eigenvectors $\tilde{T}_a$

solve the characteristic equation:

$$\det(\rho - \mathbb{1}\,\vec{\rho}) = 0$$

matrix representation of $\mathrm{ad}_X(\tilde{T}_a)$

We want that solutions always $\exists$, therefore we use the field $\mathbb{C}$ (smallest algebraic complete field) instead of $\mathbb{R}$ from now on.

Finally, we need the maximal set of generators X which simultainiously solve the eigenvalue equation.

$\rightarrow$ have to linearly independent & commute

They, $\{H_i\}$, span the abelian Cartan subalgebra G.

$$[H_i, H_j] = 0$$

The dimension of this subalgebra is called rank

$$r := \text{rank}\, G = \dim G.$$