3. The electromagnetic (spin 1) field

3.1. Complex Klein-Gordon field

Idea: combine two real scalar fields ϕ_1, ϕ_2 to one complex scalar field ϕ

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \qquad \bar{\phi}_{1/2} = \phi_{1/2}$$
$$\bar{\phi} = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$$

Result: $\mathcal{L}(\phi) = \mathcal{L}_{KG}(\phi_1) + \mathcal{L}_{KG}(\phi_2)$

$$\boxed{\mathcal{L}(\phi) = \partial_\mu \phi \, \partial_\mu \bar{\phi} - m^2 \phi \bar{\phi}}$$

remember: $\mathcal{L}_{KG} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$

Field equations: ↯ treat ϕ and $\bar{\phi}$ as independent

$$\frac{\delta \mathcal{L}}{\delta \phi} = 0 \Rightarrow \partial_\mu \partial^\mu \bar{\phi} + m^2 \bar{\phi} = 0$$

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Lagrangian $\mathcal{L}$ is invariant under

$$\phi \to e^{-i\Lambda} \phi \quad \text{and} \quad \bar{\phi} \to e^{i\Lambda} \bar{\phi}$$

Λ is a real constant.

generates Lie group $U(1)$

↓

go to the Lie algebra $u(1)$

3.2. Conserved currents and charges

infinitesimal: $\delta \phi = -i \phi \Lambda$, $\delta \bar{\phi} = i \bar{\phi} \Lambda$

$$\delta S = 0 = \int d^4x \left[\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \partial_\mu (\delta \phi) + \frac{\delta \mathcal{L}}{\delta \phi} \delta \phi + \ldots \delta \bar{\phi} \ldots \right]$$

$$= \int d^4x \, \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \delta \phi(\Lambda) + \ldots \delta \bar{\phi}(\Lambda) \ldots \right) -$$

integration by parts

$$\int d^4x \left[\underbrace{\left(\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} - \frac{\delta \mathcal{L}}{\delta \phi} \right)}_{\text{field equations for } \bar{\phi} \; = 0 \; \text{(on shell)}} \delta \phi(\Lambda) + \ldots \delta \bar{\phi}(\Lambda) \right] = 0$$

$$0 = \int d^4x \, \partial_\mu J^\mu \qquad \text{with}$$

$$J^\mu = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \delta \phi(\Lambda) + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \bar{\phi})} \delta \bar{\phi}(\Lambda)$$

$$\boxed{J^\mu = i\left(\bar{\phi}\partial^\mu \phi - \phi \partial^\mu \bar{\phi}\right)}$$

conserved current with $\partial_\mu J^\mu = 0$

under field equations !

$$0 = \int_{t_1}^{t_2} dx^0 \Big(\partial_0 \underbrace{\int d^3x \, J^0}_{Q(x^0) = Q(t)} - \underbrace{\int d^3x \, \partial_i J^i}_{=0} \Big)$$

$= 0$ for fields that vanish @ ∞

$$0 = Q(t_2) - Q(t_1) \quad \rightarrow \quad Q(t_2) = Q(t_1) \quad \text{or}$$

$$Q(t) = \text{const.}$$

$$\boxed{Q(t) = \int d^3x \, J^0 \quad \text{is conserved under time evolution}}$$

Noether's theorem: symmetries $\Longleftrightarrow$ conserved charges

3.3. Gauge symmetries

- parameter Λ of the transformation does not depend on the position in spacetime X^μ !

$\rightarrow$ global symmetry (everywhere the same transformation)

Question: Can we make it local, i.e. $\Lambda(X^\mu)$?

$$\Lambda \rightarrow \Lambda(X^\mu)$$

$$\delta \mathcal{L} = \ldots = (\partial_\mu \Lambda) J^\mu \neq 0$$

We gauge $\mathcal{L}$ (strategy: compensate with additional terms) !

$$\mathcal{L}_1 = -e J^\mu A_\mu \qquad \text{with} \qquad \boxed{\delta A_\mu = \frac{1}{e}\partial_\mu \Lambda}$$

new field called gauge field

$$\delta \mathcal{L}_1 = -e\,\delta J^\mu A_\mu - J^\mu \partial_\mu \Lambda$$

$= -e J^\mu \delta A_\mu$ term we want

but we also get this one :-c

$$\delta J^\mu = 2\bar{\phi}\phi\partial^\mu \Lambda \quad \rightarrow \quad \delta(\mathcal{L} + \mathcal{L}_1) = -2eA_\mu \partial^\mu \Lambda \phi \bar{\phi}$$

again compensate !

$$\mathcal{L}_2 = e^2 A_\mu A^\mu \phi \bar{\phi} \quad \rightarrow \quad \delta \mathcal{L}_2 = 2e A_\mu \partial^\mu \Lambda \, \phi \bar{\phi} \qquad :-)$$

$$\mathcal{L}_{\text{gauged}} = \mathcal{L} + \mathcal{L}_1 + \mathcal{L}_2$$
$$= (\partial_\mu \phi + ie A_\mu \phi)(\partial^\mu \bar{\phi} - ie A^\mu \bar{\phi}) - m^2 \phi \bar{\phi} \quad //$$

↯ symmetry of this Lagrangian is hard to see, can we do better ?

Yes ! With <u>covariant derivatives</u>:

$$\boxed{D_\mu \phi = (\partial_\mu + ie A_\mu)\,\phi} \qquad \text{with}$$

$$\delta(D_\mu \phi) = \delta(\partial_\mu \phi) + ie\, \delta A_\mu\, \phi + ie A_\mu\, \delta \phi$$
$$= -i\, \partial_\mu (\Lambda \phi) + i\cancel{e} \frac{1}{\cancel{e}} \partial_\mu \Lambda\, \phi + e A_\mu \Lambda \phi$$
$$= -i \Lambda (\partial_\mu \phi + ie A_\mu \phi)$$
$$= -i \Lambda (D_\mu \phi) \quad //$$

remember : $\delta \phi = -i \Lambda \phi$

compare : $\delta(D_\mu \phi) = -i \Lambda (D_\mu \phi)$

$D_\mu \phi$ transforms like ϕ, namely <u>covariantly</u>

$$\boxed{\mathcal{L}_{\text{gauged}} = D_\mu \phi \, \overline{D^\mu \phi} - m^2 \phi \bar{\phi}}$$

<u>Question</u>: Can we generate more covariant quantities ?

Yes ! i.e. $[D_\mu, D_\nu]\, \phi = (\partial_\mu + ie A_\mu)(\partial_\nu + ie A_\nu)\, \phi - (\mu \leftrightarrow \nu)$

$$= \cancel{\partial_\mu \partial_\nu \phi} + ie\, \partial_\mu (A_\nu \cancel{\phi}) + \cancel{ie A_\mu \partial_\nu \phi} - e^2 \cancel{A_\mu A_\nu} \phi - (\mu \leftrightarrow \nu)$$
$$= ie \underbrace{(\partial_\mu A_\nu - \partial_\nu A_\mu)}_{F_{\mu\nu}} \phi$$

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu} = \text{electromagnetic field tensor}$$

$$\delta F_{\mu\nu} = 2 \frac{1}{e} \partial_{[\mu} \partial_{\nu]} \Lambda = 0 \quad // \qquad X_{[\mu\nu]} = \tfrac{1}{2}(X_{\mu\nu} - X_{\nu\mu})$$

$$\boxed{\mathcal{L}_{\text{tot}} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi\, \overline{D^\mu \phi} - m^2 \phi \bar{\phi}}$$

kinetic term with two derivatives

3.4. The QED Lagrangian

$$\psi(x) \to e^{i\alpha(x)} \psi(x) \qquad A_\mu \to A_\mu - \frac{1}{e} \partial_\mu \alpha(x)$$

symmetry of Dirac Lagrangian for $\alpha(x) = \text{const}$.

$$D_\mu = \partial_\mu + ie A_\mu$$

$$\boxed{\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi}$$

3.5. Quatisation of A_μ

Just look @ $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

observe: $A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda$ does not change $F_{\mu\nu}$!

gauge transformation

This redundancy is a problem for quantisation !

Strategy: Remove it by gauge fixing.

(1.) Lorentz gauge $\quad \partial_\mu A'^\mu = \partial_\mu A^\mu + \partial_\mu \partial^\mu \Lambda = 0$

$$\to \boxed{\partial_\mu \partial^\mu \Lambda = -\partial_\mu A^\mu} \quad (1)$$

still not unique

(2.) Coulomb gauge $\quad A'^0 = A^0 + \underbrace{\partial^0}_{\frac{\partial}{\partial t}}\Lambda = 0$

$$\to \boxed{\frac{\partial}{\partial t} \Lambda = -A^0}$$

together with (1) we than have (after dropping the '):

$$A_0 = A^0 = 0 \quad \text{and} \quad \partial_i A^i = 0$$

reduce from 4 degrees of freedom in A_μ to 2

Quantisation:

(I.) conjugate momentum for A_μ:

$$\Pi^0 = \frac{\delta \mathcal{L}}{\delta \dot{A}_0} = 0 \,, \qquad \Pi^i \frac{\delta \mathcal{L}}{\delta \dot{A}_i} = -\dot{A}^i + \partial^i A^0 = F^{0i} = E^i$$

Electric field strength

II. cannonical commutator:

$$[A^i(\vec{x}), E^j(\vec{y})] = i \int \frac{d^3k}{(2\pi)^3} \left(\delta^{ij} - \frac{k^i k^j}{\vec{k}^2} \right) e^{i\vec{k}(\vec{x}-\vec{y})}$$

becomes the $\delta^{ij}\delta(\vec{x}-\vec{y})$ we know — new! why?

because: $\partial_i A^i = 0$

and therefore: $[\partial_i A^i(\vec{x}), E^j(\vec{y})] = \frac{\partial}{\partial x^i} [A^i(\vec{x}), E^j(\vec{y})] = 0$

EX 2.2.: requires the additional term!

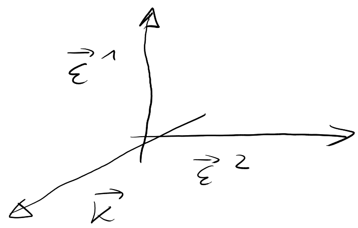
III. Mode expansion:

two remaining components (degrees of freedom) for A_μ after gauge fixing

$$\vec{A}(x) = \int \frac{d^3k}{(2\pi)^3 2k_0} \sum_{\lambda=1}^{2} \vec{\varepsilon}^{\,\lambda}(k) \left[a_k^\lambda e^{-ikx} + a_k^{\lambda\dagger} e^{ikx} \right]$$

polarisation vectors

$$\vec{\nabla}\vec{A} = \partial_i A^i = 0 \quad \rightarrow \quad \boxed{\vec{k} \cdot \vec{\varepsilon}^{\,\lambda} = 0}$$



or $\vec{\varepsilon}^{\,\lambda} \cdot \vec{\varepsilon}^{\,s} = \delta^{\lambda s}$

results in: $[a^\lambda_{\vec{k}}, a^{s\dagger}_{\vec{k}'}] = 2k_0 (2\pi)^3 \delta_{\lambda s} \delta(\vec{k}-\vec{k}')$

again creation / annihilation ops for harmonic oscillators

IV. Hamiltonian

$$H = \frac{1}{2} \int d^3x \left(\dot{\vec{A}}^2 + (\vec{\nabla} \times \vec{A})^2 \right)$$

$$\vdots$$

$$= \sum_\lambda \int \frac{d^3k}{(2\pi)^3 k_0} \frac{k_0}{2} \left(a_k^{\lambda\dagger} a_k^\lambda + \text{vacuum energy} \right)$$

vacuum energy — which we ignore!