

5.6. Quantisation of Fermions

5.6.1. Grassmann numbers

remember 2.4.: for Dirac field $\{\hat{a}_{\vec{p}}^r, \hat{a}_{\vec{q}}^{s\dagger}\} = (2\pi)^3 \delta(\vec{p}-\vec{q}) \delta^{rs}$
anti-commutator

Fine for operators, but in path integral we have just ordinary numbers (like Φ vs. ϕ from last two lectures).

Question: What shall we do there to implement fermionic fields?

Answer: Grassmann numbers?

$$\Theta \eta = -\eta \Theta$$

Implies that $\Theta^2 = 0$ and that any function $f(\Theta)$ can be written as

$$f(\Theta) = A + \Theta B \quad (\text{Taylor expansion})$$

We further need: 1) Integration $\int d\Theta f(\Theta)$

$$= \int d\Theta (A + \Theta B) \text{ should be invariant under}$$

$$= \int d\Theta ([A + B\eta] + \Theta B) \stackrel{\Theta \rightarrow \Theta + \eta}{=} B$$

$$\boxed{\int d\Theta (A + B\Theta) = B}$$

$$\text{multiple integrals: } \int d\Theta \int d\eta \eta \Theta = 1$$

$$2) \text{ Differentiation } \frac{\partial}{\partial \Theta} f(\Theta) = \frac{\partial}{\partial \Theta} (A + B\Theta) = B$$

for Grassmann numbers integration = differentiation

3) complex Grassmann numbers

$$\Theta = \frac{\Theta_1 + i\Theta_2}{\sqrt{2}}, \quad \Theta^* = \frac{\Theta_1 - i\Theta_2}{\sqrt{2}}$$

4) complex conjugation $(\Theta \eta)^* = \eta^* \Theta^* = -\Theta^* \eta^*$
 we can now evaluate the Grassmann version of a Gaussian integral

$$I = \int d\Theta^* d\Theta e^{-\Theta^* b \Theta} \xrightarrow{\text{Taylor expand}} \int d\Theta^* d\Theta (1 - \Theta^* b \Theta)$$

$$= \int d\Theta^* d\Theta (1 + \Theta b \Theta^*) = b //$$

remember: if Θ would be $\in \mathbb{C}$, $I = \frac{2\pi}{b}$

$$\int d\Theta^* d\Theta \Theta \Theta^* e^{-\Theta^* b \Theta} = 1 = \frac{1}{b} \cdot b$$

now in N dimensions $I = \left(\prod_i \int d\Theta_i^* d\Theta_i \right) e^{-\Theta_i^* B_{ij} \Theta_j}$

I) diagonalise B_{ij} with unitary transformation

$$\Theta_i' = U_{ij} \Theta_j$$

$$\prod_i \Theta_i' = \frac{1}{N!} \varepsilon^{ij\dots l} \Theta_i' \Theta_j' \dots \Theta_l' = \dots = (\det U) \left(\prod_i \Theta_i \right)$$

$$\leadsto I = \left(\prod_i \int d\Theta_i'^* d\Theta_i' \right) e^{-\sum_i \Theta_i'^* b_i \Theta_i} = \prod_i b_i = \det \underline{B} //$$

$$\boxed{\left(\prod_i \int d\Theta_i'^* d\Theta_i' \right) e^{-\Theta_i'^* B_{ij} \Theta_j} = \det \underline{B}}$$

5.6.2. Dirac Propagator

remember Dirac spinor $\hat{\Psi} = (\hat{\Psi}_1 \hat{\Psi}_2 \hat{\Psi}_3 \hat{\Psi}_4)^T$; now it becomes

$$\Psi(x) = \sum_i \psi_i \phi_i(x)$$

Grassmann number ψ_i ordinary Dirac spinor basis

$$\text{i.e.: } \phi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \dots, \phi_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

we can now evaluate:

$$\langle 0 | T \Psi(x_1) \bar{\Psi}(x_2) | 0 \rangle = \frac{\int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi e^{iS} \Psi(x_1) \bar{\Psi}(x_2)}{\int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi e^{iS}}$$

$$\text{EX} \left\{ \begin{array}{l} \text{with } S = \int d^4x \bar{\Psi} (i \not{\partial} - m) \Psi \\ \langle 0 | T \Psi(x_1) \bar{\Psi}(x_2) | 0 \rangle = S_F(x_1 - x_2) = \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-i k (x_1 - x_2)}}{k - m + i\epsilon} \end{array} \right.$$

Note: higher correlation functions are obtained from Wick's theorem

5.6.3. Generating Function

$$Z[\bar{\eta}, \eta] = \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \exp \left[i S + \int d^4x (\bar{\eta} \Psi + \bar{\Psi} \eta) \right]$$

Grassmann valued source field

$$= Z_0 \cdot \exp \left[- \int d^4x d^4y \bar{\eta}(x) S_F(x-y) \eta(y) \right]$$

and therefore

$$\langle 0 | T \Psi(x_1) \bar{\Psi}(x_2) | 0 \rangle = Z_0^{-1} \left(-i \frac{\delta}{\delta \bar{\eta}(x_1)} \right) \left(i \frac{\delta}{\delta \eta(x_2)} \right) Z[\bar{\eta}, \eta] \Big|_{\eta, \bar{\eta} = 0}$$

5.7. Symmetries in the Path Integral

three point correlator for free scalar theory

$$\langle 0 | T \phi_1 \phi_2 \phi_3 | 0 \rangle = Z_0^{-1} \int \mathcal{D}\phi e^{iS} \phi_1 \phi_2 \phi_3$$

$$S = i \int d^4x \left[\frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 \right]$$

now shift $\phi(x) \rightarrow \phi'(x) = \phi(x) + \epsilon(x)$; leaves $\mathcal{D}\phi$ invariant, $\mathcal{D}\phi = \mathcal{D}\phi'$

$$0 = \int \mathcal{D}\phi e^{iS} \left[i \int d^4x \epsilon (-\partial^2 - m^2) \phi \cdot \phi_1 \phi_2 \phi_3 + \epsilon_1 \phi_2 \phi_3 + \phi_1 \epsilon_2 \phi_3 + \phi_1 \phi_2 \epsilon_3 \right]$$

$$(\partial^2 + m^2) \langle 0 | T \phi \phi_1 \phi_2 \phi_3 | 0 \rangle = -i \delta(x - x_1) \langle 0 | T \phi_2 \phi_3 | 0 \rangle + \text{cycl.}$$

$$\text{in particular: } (\partial^2 + m^2) \langle 0 | T \phi \phi_1 | 0 \rangle = -i \delta(x - x_1)$$

We see: Feynman propagator is the Green's function of the classical field equations.

This idea generalises to arbitrary Lagrangians $\mathcal{L}$ (as long as the measure is preserved under shifts)

$$\left\langle \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \right] \phi_1 \dots \phi_n \right\rangle = \sum_{i=1}^n \left\langle \phi_1 \dots (i\delta(x-x_i)) \dots \phi_n \right\rangle$$

Schwinger - Dyson equation

contact term