

Now we look at the complex scalar field with

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \bar{\phi} - m^2 \phi \bar{\phi}$$

and $\phi(x) \rightarrow \phi'(x) = \phi(x) + i\alpha(x) \phi(x)$

Again the measure does not change, $\mathcal{D}\phi = \mathcal{D}\phi'$
repeating the steps above, we obtain

$$\langle \partial_\mu j^\mu(x) \phi_1 \bar{\phi}_2 \rangle = (-i) \langle i \phi_1 \delta(x-x_1) \bar{\phi}_2 - i \bar{\phi}_1 \bar{\phi}_2 \delta(x-x_2) \rangle$$

and $j^\mu = i(\phi \partial^\mu \bar{\phi} - \bar{\phi} \partial^\mu \phi)$

for a general Lagrangian and symmetry

$$\boxed{\langle \partial_\mu j^\mu \phi_1 \phi_2 \rangle = (-i) \underbrace{\langle \delta \phi_1 \delta(x-x_1) \phi_2 + \phi_1 \delta \phi_2 \delta(x-x_2) \rangle}_{\text{contact term}}}$$

5.8. The Electromagnetic Field

We encountered the em field in lecture 2 but did not compute its propagator yet. \rightarrow today with path integral

challenge: gauge symmetry requires care in identifying propagation (physical) degrees of freedom

$$\begin{aligned} S &= -\frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} \int d^4x A_\mu (g^{\mu\nu} \partial_\nu^2 - \partial^\mu \partial^\nu) A_\nu \\ &= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{A}_\mu(k) (-k^2 g^{\mu\nu} + k^\mu k^\nu) \tilde{A}_\nu(-k) \end{aligned}$$

↑ Fourier transformation

$$S = 0 \quad \text{for} \quad \tilde{A}_\mu(k) = k_\mu \alpha(k) \quad \rightsquigarrow \text{path integral}$$

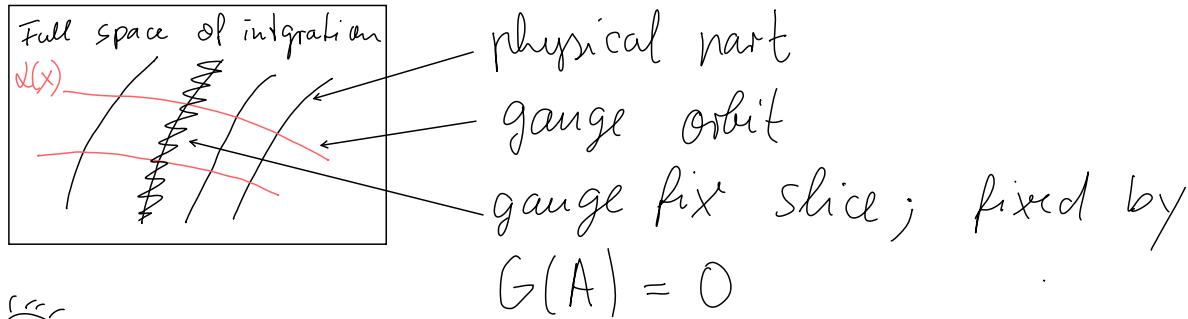
$$Z_0 = \int \mathcal{D}A e^{iS[A]} \text{ diverges}$$

or equally: $(-k^2 g_{\mu\nu} + k_\mu k_\nu) \tilde{D}_F^{\nu\sigma} = i \delta_{\mu\sigma}$

see last lecture

cannot be solved. Reason gauge symmetry (remember)
gauge transformation: $A_\mu(x) \rightarrow A_\mu(x) + \underbrace{\frac{1}{e} \partial_\mu \alpha(x)}_{\sim \partial_\mu \alpha(x) \text{ or } k_\mu \alpha(k) \text{ after FT}}$

Idea of Faddeev & Popov:



Idea: insert S-function into the path integral which restricts to gauge fixed slice

1) Define path integral version of S-function.

$$1 = \int D\alpha(x) S(G(A^\alpha)) \det \left(\frac{\delta G(A^\alpha)}{\delta \alpha} \right)$$

$$A^\alpha(x) = A_\mu(x) + \frac{1}{e} \partial_\mu \alpha(x) \quad \nearrow EX$$

originates from $1 = \left(\prod_i \int da_i \right) S^{(n)}(\vec{g}(\vec{a})) \det \left(\frac{\partial g_i}{\partial a_j} \right)$

$$Z_0 = \det \left(\frac{\delta G(A^\alpha)}{\delta \alpha} \right) \underbrace{\int D\alpha \underbrace{\int D\alpha e^{iS[A]} S(G(A))}_{\text{divergent physical part}}}_{\text{divergent physical part}}$$

with $G(A) = \partial^\mu A_\mu(x) - w(x) \leftarrow \text{any scalar function}$

$$Z_0 = \det \left(\frac{1}{e} \partial^2 \right) \left(\int D\alpha \right) \int D\alpha e^{iS[A]} S(\partial^\mu A_\mu - w(x))$$

2) Average over gauge choices $w(x)$

$$Z_0' = N(\xi) \int D\omega \exp \left[-i \int d^4x \frac{\omega^2}{2\xi} \right] \cdot Z_0$$

Gauss function
centred at $\omega(x)=0$

$$= N(\xi) \det \left(\frac{1}{\epsilon} \partial^2 \right) \left(\int D\omega \right) \int D\mathbf{A} e^{iS[A]} \exp \left[-i \int d^4x \frac{1}{2\xi} (\partial^\mu A_\mu)^2 \right]$$

normalisation factor

Correlation functions:

$$\langle 0 | T \mathcal{O}(A) | 0 \rangle = \frac{\int D\mathbf{A} \mathcal{O}(A)}{Z_0} e^{iS}$$

i · S_{eff}

gauge invariant

$$= \frac{\int D\mathbf{A} \mathcal{O}(A) \exp \left[i \left(S - \frac{1}{2\xi} \int d^4x (\partial^\mu A_\mu)^2 \right) \right]}{\int D\mathbf{A}}$$

All inconvenient factors cancel!

for S_{eff} $\left[-K^2 g_{\mu\nu} + (1-\xi^{-1}) K_\mu K_\nu \right] \tilde{D}_F^{\mu\nu}(k) = i \delta_{\mu\nu}$
 which is solved by

$$\boxed{\tilde{D}_F^{\mu\nu}(k) = \frac{-i}{k^2 + i\varepsilon} \left(g^{\mu\nu} - (1-\xi) \frac{k^\mu k^\nu}{k^2} \right)}$$

in particular $\xi=0$ Landau gauge

$\xi=1$ Feynman gauge \leftarrow we use this one

S.9. Non-abelian Gauge Fields

Similar derivation with

- $(A^\alpha)_\mu^a = A_\mu^a + \frac{1}{g} \partial_\mu \alpha^a + f_{bc}^a A_\mu^b \alpha^c = A_\mu^a + \frac{1}{g} D_\mu^a \alpha^\alpha$
- $G(A) = \partial^\mu A_\mu^a(x) - \omega^a(x)$

propagator: $\tilde{D}_F^{\mu\nu;ab}(k) = \tilde{D}_F^{\mu\nu}(k) \delta^{ab}$

$\xi=1$ is now called Feynman-'t Hooft gauge

$$\frac{\delta G(A^\alpha)}{\delta \alpha} = \frac{1}{g} \gamma^\mu D_\mu \text{ now depends on } A_\mu$$

$\Rightarrow \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right)$ cannot be factored out, instead

$$\int \mathcal{D}c \mathcal{D}\bar{c} \exp\left[i \int d^4x \bar{c} \not{\partial}^\mu c\right]$$

have to be Grassmann numbers

on the other hand c behaves like a complex scalar field ($\text{spin} = 0$) \rightarrow

c violates spin-statistic theorem

Faddeev-Popov ghosts

$$L_{\text{eff}} = -\frac{1}{4}(F_{\mu\nu}^\alpha)^2 + \frac{1}{2g}(\not{\partial}^\mu A_\mu^\alpha)^2 - \bar{c}^\alpha \not{\partial}^\mu D_\mu c^\alpha$$

ghost = wrong sign for kinetic term

Feynman rules in Exercise?