2. Mathematical foundations

course: ideas from $SO(3)$ example $\xrightarrow{\text{generalise}}$ much more complicated Lie groups & algebras

But first, define the objects we are working with properly!

2.1. Group

Def: A group is a set G with an operation, called multiplication, $\circ$ such that:

1) $g_1, g_2 \in G \Rightarrow g_1 \circ g_2 \in G$ closure
2) $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3 \quad g_1, g_2, g_3 \in G$ associativity
3) $\exists\, e \in G$ such that $g = e \circ g = g \circ e$ $\forall\, g \in G$ existence of identity
4) $\forall\, g \in G,\ \exists\, g^{-1}$ such that $g \circ g^{-1} = g^{-1} \circ g = e$ existence of inverse

If in addition:

5) $g \circ g' = g' \circ g \quad \forall\, g, g' \in G$ commutative

holds, the group is called abelian.

Examples:
- permutation group S_n of n ordered elements has $n! = 1 \cdot 2 \cdot \ldots \cdot (n-1) \cdot n$ group elements.
  $\rightarrow$ finite group
- $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ $\quad a, b \in \mathbb{Z}_n \quad a \circ b = (a+b) \bmod n$

identity: $e = 0$
inverse: $(a^{-1} + a) \bmod n = e = 0$

group inverse, not $a^{-1} = \frac{1}{a}$

$a^{-1} = (-a) \bmod n$

check: $a \in \mathbb{Z}_n \rightarrow a^{-1} \in \mathbb{Z}_n$ ✓

Note: A smallest set of $\{g_1, \ldots, g_N\} \in G$ such that any element $g \in G$ can be obtained as product of them is called a basis of the group. Its elements are called generators.

2.2. Field

A *field* $\mathbb{F}$ is a set with two operations: addition $+$ and scalar multiplication $\cdot$ such that:

1) $\mathbb{F}$ is an abelian group under $+$ with "0" as identity element
2) $\mathbb{F} - \{0\}$ is a group under multiplication with "1" as identity
3) $a,b,c \in \mathbb{F} \Rightarrow$ $a\cdot(b+c) = a\cdot b + a\cdot c$; $(a+b)\cdot c = a\cdot c + b\cdot c$ distributivity

If additionally:

4) $a\cdot b = b\cdot a$

the field is called commutative

relevant in physics are:

- real numbers $\mathbb{R} \ni a,b$
- complex numbers $\mathbb{C} \ni c = a + i b$, $\sqrt{-1}$, $i^2 = -1$
- quaternions $\mathbb{H}$

$$i^2 = j^2 = k^2 = -1 \qquad \begin{aligned} i\cdot j &= -j\cdot i = k \\ j\cdot k &= -k\cdot j = i \\ k\cdot i &= -i k = j \end{aligned}$$

$\hookrightarrow$ $\mathbb{H} \ni q = a + ib + jc + kd \qquad a,b,c,d \in \mathbb{R}$
$\qquad = (a+ib) + (c+id)j$

4 real = 2 complex parameters

↯ quaterions are non-commutative

Lorentzgroup

SO(3), SO(3,1)
SO(3,1,$\mathbb{R}$)

SL(2,$\mathbb{R}$) real 2x2 matrices + det $M = 1$

SL(2,$\mathbb{C}$) complex 2x2 matrices + det $M = 1$

SL(2,$\mathbb{C}$) = SO(3,1)

<u>complex conjugation:</u>

$\mathbb{C}$: $(1, i)^* = (1^*, i^*) = (1, -i)$

$\mathbb{H}$: $(1,i,j,k)^* = (1^*, i^*, j^*, k^*) = (1, -i, -j, -k)$

sometimes we also use $\bar{\ }$ instead $*$.

2.3. Vector space

A <u>vector space</u> V over a field $\mathbb{F}$ satisfies:

1) V is an abelian group under addition $+$
2) $a \in \mathbb{F}$, $v \in V$ $\Rightarrow$ $a \cdot v \in V$ — closure

   scalar multiplication

3) $a \cdot (v+w) = a \cdot v + a \cdot w$ $\quad v, w \in V$ — bilinearity

   $(a+b)\, w = a \cdot w + b \cdot w$ $\quad a, b \in \mathbb{F}$

- Elements $v \in V$ are called <u>vectors</u>
- A <u>basis</u> B for V is a <u>minimal</u> set of N vectors $e_i \in V$ such that any vetor $v \in V$ can be represented as:

$$v = \sum_i a_i e_i \qquad \text{with } a_i \in \mathbb{F}$$

- The number N of basis vectors is called <u>dimension</u> of V:

$$\dim_{\mathbb{F}}(V) = N$$

depends on the field we use

in particular we have

$$4 \dim_{\mathbb{H}}(V) = 2 \dim_{\mathbb{C}}(V) = \dim_{\mathbb{R}}(V)$$

Combining Vector space

I. <u>direct sum</u> $\quad V_1 \oplus V_2$, both over the same field $\mathbb{F}$

defined by:

1) $a \cdot (v_1 \oplus v_2) = (a \cdot v_1) \oplus (a \cdot v_2) \qquad a \in \mathbb{F}$
2) $v_1 \oplus v_2 + w_1 \oplus w_2 = (v_1 + w_1) \oplus (v_2 + w_2) \qquad v_1, w_1 \in V_1, \quad v_2, w_2 \in V_2$

in practice: $\quad V = V_1 \oplus V_2 = \begin{pmatrix} v_{1i} \\ v_{2j} \end{pmatrix} \qquad \vec{v}_1 = \begin{pmatrix} v_{11} \\ \vdots \\ v_{1n} \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} v_{21} \\ \vdots \\ v_{2m} \end{pmatrix}$

II. <u>cartesian product</u>: $\quad V_1 \otimes V_2$, again over the same field $\mathbb{F}$

is the set of tuples (v_1, v_2) which satisfy: $\quad v_1 \in V_1,\ v_2 \in V_2$

1) $(a v_1, v_2) = (v_1, a v_2) \qquad a \in \mathbb{F}$
2) $(v_1 + w_1, v_2 + w_2) = (v_1, v_2) + (v_1, w_2) + (w_1, v_2) + (w_1, w_2)$

or in practice: $\quad V_{ij} = V_{1i} \cdot V_{2j}$

for the dimensions

$$\dim_{\mathbb{F}}(V_1 \oplus V_2) = \dim_{\mathbb{F}}(V_1) + \dim_{\mathbb{F}}(V_2)$$

$$\dim_{\mathbb{F}}(V_1 \otimes V_2) = \dim_{\mathbb{F}}(V_1)\,\dim_{\mathbb{F}}(V_2)$$

holds. $\quad 3 \otimes 3 \to 1 \oplus 3 \oplus 5$

2.4 Linear algebra

A linear algebra A consists of a vector space A over a field $\mathbb{F}$ with an additional vector multiplication $\times$ such that:

1) $v, w \in A \rightarrow v \times w \in A$ closure
2) $(v_1 + v_2) \times w = v_1 \times w + v_2 \times w$ bilinearity
   $v \times (w_1 \times w_2) = v \times w_1 + v \times w_2$

2.5. Lie algebra

A Lie algebra g is a linear algebra with the Lie bracket $[\cdot,\cdot]$ as vector product, satisfying:

1) $[x, y] = -[y, x] \quad \forall\, x, y \in g$
2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ Jacobi identity

$$\frac{d}{dx}(f \cdot g) = \left(\frac{d}{dx} f\right) \cdot g + f \cdot \left(\frac{d}{dx} g\right) \quad \text{Leibnizrule}$$

$$[z, \cdot\,] \sim \frac{d}{dx} \cdot$$

$$[z, [x, y]] = [[z, x], y] + [x, [z, y]]$$

Remarks: - if the elements of g can be realised as $n \times n$ matrices A, B with

$$[A, B] = A \cdot B - B \cdot A$$

then 1) and 2) hold automatically

- if we do not have matrices:

$$[T_a, T_b] = f_{ab}{}^c\, T_c$$

($f_{ab}{}^c$: structure coefficients)

1) $f_{ab}{}^c = -f_{ba}{}^c$

2) $\sum_c \left(f_{ab}{}^c f_{cd}{}^e + f_{bd}{}^c f_{ca}{}^e + f_{da}{}^c f_{cb}{}^e \right) = 0$

2.6* Lie group

A Lie or continous group is an n-dimensional manifold G and a mapping $\varphi : G \times G \to G$ such that φ defines a group multiplication.

The mappings φ and $\psi : G \to G$, defined as $\psi(a) = a^{-1}$ inverse element are both continous.