Tutorial for String Theory I, WiSe2013/14
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10. Vertex operators and the complex plane

To be discussed on Thursday, January 9, 2013 in the tutorial.

Exercise 10.1: Vertex operators, OPE's and correlation functions

a) Compute
$$: e^{ik_\mu X^\mu} : (z, \bar{z}) X(0) \,.$$

b) Compute
$$: e^{ik_\mu X^\mu} : (z, \bar{z}) \partial X^\nu(0) \,.$$
Hint: This OPE can be computed in two ways: either (i) directly or (ii) by using 1 a).

c) Show that
$$: e^{ik_\mu X^\mu} : (z, \bar{z})$$
is a primary field with conformal wieghts $(\alpha' k^2/4\,,\ \alpha' k^2/4)$.
Hint: Calculate its OPE with T.

d) Compute
$$: e^{ik_\mu X^\mu} : (z, \bar{z}) : e^{ik_\mu X^\mu} : (0)$$

e) Compute
$$\big\langle : e^{ik_1 \cdot X} : (z_1, \bar{z}_1) : e^{ik_2 \cdot X} : (z_2, \bar{z}_2) : e^{ik_3 \cdot X} : (z_3, \bar{z}_3) \big\rangle$$
appearing in the closed string tree amplitude for 3 tachyons.

Exercise 10.2: Asymptotic in and out states

A (chiral) primary field $\phi(z)$ with conformal weight h in the complex plane has the following mode expansion:
$$\phi(z) = \sum_{n \in \mathbb{Z}} z^{-n-h} \phi_n \,. \tag{1}$$
As $z \to 0$ corresponds to $\tau \to -\infty$ on the cylinder, a state of the form
$$|\phi_{\text{in}}\rangle := \lim_{z \to 0} \phi(z) |0\rangle$$
can be considered as asymptotic "in-state". For this state to be non-singular, we need to impose (cf. eq. (1))
$$\phi_n |0\rangle = 0 \quad \forall n \geq 1 - h \,. \tag{2}$$

a) Verify that, if (1) is satisfied, we have
$$|\phi_{\text{in}}\rangle = \phi_{-h} |0\rangle$$

with

$$\phi_{-h} = \oint\limits_{C_0} \frac{dz}{2\pi i} \frac{\phi(z)}{z}$$

being the coefficients ϕ_n for $n = -h$.

b) Using

$$[L_n, \phi_m] = [n(h-1) - m]\phi_{n+m}$$

and

$$L_n|0\rangle = 0 \quad \forall n \geq -1\,,$$

show

$$L_0|\phi_{\text{in}}\rangle = h|\phi_{\text{in}}\rangle$$
$$L_n|\phi_{\text{in}}\rangle = 0\,, \quad n > 0\,,$$

i.e., that $|\phi_{\text{in}}\rangle$ is a highest weight state of a representation of the Virasoro algebra (alias "Verma module").

Remark: The primary operators of a CFT are obviously in one-to-one correspondence with highest weight states of Verma modules. As was explained in lecture, a full Verma module is generated by acting with $L_{-n}(n > 0)$ on a lowest weight state. The states so-generated are called descendant states. They can be generated directly from the vacuum by acting with descendant fields, which are operators that occur in (possibly multiple) operator products of the primary field with the energy momentum tensor.

c) For a non-chiral primary field $\phi(z, \bar{z})$ with conformal weights $(h, \bar{h})$, one defines in a similar way:

$$|\phi_{\text{in}}\rangle = \lim_{z,\bar{z}\to 0} \phi(z, \bar{z})|0\rangle = \phi(0, 0)|0\rangle\,,$$

which results in

$$\overset{(-)}{L}_0|\phi_{\text{in}}\rangle = \overset{(-)}{h}\,|\phi\rangle$$
$$\overset{(-)}{L}_n|\phi_{\text{in}}\rangle = 0 \quad \forall n > 0\,.$$

From this, read off what extra condition has to be imposed on the primary operator $\phi(z, \bar{z})$ for the state $|\phi_{\text{in}}\rangle$ to be physical in the sense of the old covariant quantization of the close string. (I.e., we assume that $\phi(z, \bar{z})$ is a primary operator in the conformal field theory that describes the quantized closed bosonic string).

Remark: Primary operators that satisfy the additional constraint to be found in part c) are called vertex operators. They can be used to create physical in and out states from the vacuum. In string theory, scattering amplitudes of asymptotic in out out states are thus determined by vacuum expectation values of products of the vertex operators that correspond to those in and out states. The vertex operator for the close string tachyon, for example, is given by $: e^{ik_\mu X^\mu(z,\bar{z})} :$ with $k^2 = 2$, whereas the graviton, Kalb-Ramond field and dilaton corresponds to particular linear combinations of the vertex operators $: \partial X^\mu(z)\bar{\partial}\bar{X}^\nu(\bar{z})e^{ik_\rho X^\rho(z,\bar{z})} :$ with $k^2 = 0$. To verify that these are primary operators and that they satisfy the additional requirement of part c), one has to calculate the operator product with the (normal-ordered) energy momentum tensor using Wick's theorem for products of normal-ordered operators (cf. the lecture).

Exercise 10.3: The complex plane and the cylinder

A special case of a conformal/holomorphic transformation is the map

$$z' \to z = e^{z'} , \quad \bar{z}' \to \bar{z} = e^{\bar{z}'} ,$$

which maps the cylinder (i.e., the Wick-rotated world sheet of a non-interacting closed string) to the complex plane.

a) Calculate the rescaling function $f(z, \bar{z})$ introduced in Problem 7.1 c) (on the last exercise sheet) for this conformal transformation.

b) What is the image of a curve of constant τ under this transformation?

c) Determine how σ transformations $\sigma \to \sigma + \theta$ and time translations $\tau \to \tau + a$ operate on the new coordinates z and $\bar{z}$ on the complex plane and interpret the result geometrically.