Tutorial for String Theory I, WiSe2013/14
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# 9. Conformal Field Theory

To be discussed on Thursday, December 19, 2013 in the tutorial.

**Exercise 9.1: Conformal transformations as area preserving maps**

In general, a conformal transformation $x \to \tilde{x}(x)$ is defined to be a transformation that preserves the metric up to a local scale factor:

$$\tilde{g}_{pq}(\tilde{x}(x)) = \Omega^2(x) \frac{\partial x^m}{\partial \tilde{x}^p} \frac{\partial x^n}{\partial \tilde{x}^q} g_{mn}(x) \,.$$

Specializing from now on to positive definite curved metric (i.e., Euclidean signature) the angle $\alpha$ between two vector fields $v^m(x)$ and $w^m(x)$ at a point $x_0$ is defined by

$$\cos\alpha(v,w)(x_0) := \left. \frac{v^m w^n g_{mn}}{||v||\,||w||} \right|_{x=x_0} \,.$$

Here, $||v|| := \sqrt{v^m v^n g_{mn}}$ is the length, or norm, of a vector $v^m$.

a) Show that a conformal transformation is angle-preserving, i.e., that

$$\cos\alpha(\tilde{v},\tilde{w})(\tilde{x}(x_0)) = \cos\alpha(v,w)(x_0) \,,$$

where

$$\hat{v}^m(\tilde{x}(x)) = \frac{\partial \tilde{x}^m}{\partial x^n} v^n(x)$$

is the transformed vector field.

b) In conformal gauge, the 2D Lorentzian world sheet metric is

$$\begin{aligned} ds^2 &= \Omega^2(\sigma,\tau)(-d\tau^2 + d\sigma^2) \\ &= -\Omega^2(\sigma^+,\sigma^-) d\sigma^+ d\sigma^- \,. \end{aligned}$$

Performing the Wick rotation

$$\sigma^\pm = (\tau \pm \sigma) \to -i(\tau \pm i\sigma) \,,$$

write down the resulting Euclidean metric both in terms of the (Wick-rotated) $(\tau,\sigma)$ and the complex coordinates

$$z' = \tau - i\sigma \,, \quad \bar{z}' = \tau + i\sigma \,.$$

c) Show that all holomorphic coordinate transformations

$$z' \to \tilde{z}(z') \,, \quad \bar{z}' \to \bar{\tilde{z}}(\bar{z}')$$

change the metric only by a local rescaling $\Omega^2(z',\bar{z}') \to f(z',\bar{z}')\Omega^2(z',\bar{z}')$, i.e., that they are conformal.

**Exercise 9.2: Fractional linear transformation**

The group $\mathrm{SL}(2,\mathbb{R})$ of $(2\times 2)$-matrices of unit determinant acts on the Riemann-sphere (i.e. on $\mathbb{C}\cup\{\infty\}$) by so-called fractional linear transformations:

$$z \to z' = \frac{az+b}{cz+d}\,,$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})\,. \tag{1}$$

a) Show that two successive fractional linear transformations,

$$z \to z' = \frac{az+b}{cz+d}\,, \quad z' \to z'' = \frac{ez'+f}{gz'+h}\,,$$

are equivalent to one fractional linear transformation

$$z \to z'' = \frac{jz+k}{lz+m}\,,$$

where the matrix

$$\begin{pmatrix} j & k \\ l & m \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$$

is the product of two $\mathrm{SL}(2,\mathbb{R})$ matrices that correspond to the single transformations $z \to z'$ and $z' \to z''$.

b) Show that the fractional linear action of the inverse matrix of (1) on $z'$ leads back to $z$, and hence corresponds to the inverse transformation $z' \to z$.

**Exercise 9.3: Normal ordering**

Compute the following conformal normal ordered product for a free boson $X^\mu$.

a) $:X^\mu(z_1,\bar z_1)X_\mu(z_2,\bar z_2)X^\nu(z_3,\bar z_3)X_\nu(z_4,\bar z_4):$

b) $:\partial X^\mu(z_1,\bar z_1)\partial X^\nu(z_2,\bar z_2):$

c) $:\partial X^\mu(z_1,\bar z_1)\bar\partial X^\nu(z_2,\bar z_2):$

**Exercise 9.4: Operator product expansion**

Compute the singular terms of the following OPE's.

a) $\partial X^\mu(z)\partial X^\nu(0)$

b) $\partial X^\mu(z)\bar\partial X^\nu(0)$