Lie Algebras and Lie Groups

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- lecture notes and notebooks on
  https://www.fhassler.de/teaching#Lie_2022
- lectures on Monday 12:15 – 14:00
  28.02. – 28.03. online on MS Teams
  afterwards in person? room 447
- oral exam planned for end of semester
  ⚡ attendace and active participation in the lectures are crucial to pass the exam!
- Mathematica is used for computations
  - a) student license 800 zł
    for on semester 250 zł
  - b) check for university licenses
    i.e. computer lab of Institute of Mathematics
  - c) free Wolfram Engine for Developers
    + Jupyter Notebook
    install instructions are on the course website
- Course is based on a course by
  Prof. Stefan Groot Nibbelink
  held in 2014 at LMU Munich

↗ Literature at the website

1. Introduction and Motivation

1.1. Why do we care?

- (Lie) groups characterise symmetries in physical systems

Classical mechanics: - help to solve equations of motion

quantum mechanics: - quantisation of angular momentum
$\Rightarrow$ Lie group $SU(2)$
- quantum numbers l and m

quantum field theory:

- Lorentz group and Poincaré group
  $\Rightarrow$ spin of particles
- gauge group characterise particle content of the standard model

- Lie Groups parameterised by continuous set of variables = coordinates on manifold

tangent space = Lie algebra

- encodes already most aspects of the Lie group
- much easier to deal with (just linear algebra)

$\Rightarrow$ complete classification of semi simple Lie algebras

later in the course

- Lie groups / algebras are not directly visible in physics
  $\Rightarrow$ see only representations
  i.e. Lorentz group, $SO(3,1)$, acts
  - on scalars (trivially) ϕ ← Higgs boson
  - on vector V^μ ← boson, gauge potential i.e. photon
  - on spinor ψ
  - ...

representations decompose into fundamental building blocks

= irreducible repr. = irreps

1.2. Rotations in 3 dim. as example

defined by: $R^T \cdot R = \mathbb{1}_3$, $\det R = 1$ $\rightsquigarrow SO(3)$

$R \in M_{3\times 3}(\mathbb{R})$, real 3×3 matrices examples:

$$R_1(\alpha_1)\vec{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_1 & \sin\alpha_1 \\ 0 & -\sin\alpha_1 & \cos\alpha_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$R_2(\alpha_2) = \begin{pmatrix} \cos\alpha_2 & 0 & \sin\alpha_1 \\ 0 & 1 & 0 \\ -\sin\alpha_2 & 0 & \cos\alpha_2 \end{pmatrix}$$

$$R_3(\alpha_3) = \begin{pmatrix} \cos\alpha_3 & \sin\alpha_3 & 0 \\ -\sin\alpha_3 & \cos\alpha_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

a general rotation can be written as:

$$R = R_1(\alpha_1)\, R_2(\alpha_2)\, R_3(\alpha_3)$$

check: 3×3 matrix R has $3\cdot 3 = 9$ real parameters

$$(R^T\cdot R)^T = (\mathbb{1}_3)^T = R^T R = \mathbb{1}_3$$

$\hookrightarrow \hat{=} \frac{3\cdot(3+1)}{2} = 6$

$\det R = 1$ $\qquad \det(R^T\cdot R) = \det(\mathbb{1}_3) = 1 = \det R^T \det R$

$= (\det R)^2$

$\rightsquigarrow \det R = \pm 1$

discrete choice

$$\underbrace{9}_{\text{parameters}} - \underbrace{6}_{\text{constraints}} = 3 \longleftarrow \text{dim of Lie group } SO(3)$$

This group is called $\underset{\text{special orthogonal group}}{SO(3)} \overset{\det R = 1}{} \subset O(3)$, $\det R = \pm 1$

1.2.1. so(3) Lie algebra

tangent space of $SO(3)$ at the identity element

$$\mathbb{1}_3 = R(0,0,0) = R(\vec{0})$$

spanned by $\boxed{E_i = \frac{\partial}{\partial\alpha_i} R(\vec{\alpha})\Big|_{\vec{\alpha}=\vec{0}}}$

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

They span the vector space of antisymmetric, real 3×3 matrices

Check: $E^T = -E$ $\frac{1}{2}3(3+1) = 6$ constraints

$9-6 = 3$ dimensional

Lie algebra = vector space V ✓
+ product $V\times V \to V$

naive product would be matrix product

$$E_1 \cdot E_2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{not antisymmetric}$$

correct product is the commutator

$$[E_i, E_j] = E_i \cdot E_j - E_j \cdot E_i = \sum_{k=1}^{3} \varepsilon_{ijk} E_k$$

Levi-Civita symbol, totally antisymmetric with $\varepsilon_{123} = 1$

$$[E_1, E_2] = \overset{=1}{\varepsilon_{123}} E_3$$
$$[E_3, E_1] = \varepsilon_{312} E_2 = \overset{=1}{\varepsilon_{123}} E_2$$
$$[E_2, E_3] = \varepsilon_{231} E_1 = \underset{=1}{\varepsilon_{123}} E_1$$

1.2.2. Outlook representations

Can we find other matrices E_i' with the same commutators?
real

Yes! i.e. the product representation

$$E_i' = E_i \otimes \mathbb{1}_3 + \mathbb{1}_3 \otimes E_i$$
Kronecker product

(QM: couple angular momentum
$so(3) = su(2)$

with the Kronecker product

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{pmatrix} \qquad \begin{matrix} A \in M_{n\times n} \\ B \in M_{m\times m} \end{matrix}$$

$A \otimes B \in M_{n\cdot m \times n\cdot m} \quad \Rightarrow \quad E_i' \in M_{9\times 9 = 3\cdot 3}$

we can check : $[E_i', E_j'] = [E_i, E_j] \otimes \mathbb{1}_3 + \mathbb{1}_3 \otimes [E_i, E_j]$

$$= \sum_{k=1}^{3} \varepsilon_{ijk} E_k'$$

<u>But</u> not an irrep ; use Casimir operator

$$C := -\underset{\substack{\shortparallel \\ E_1'\cdot E_1'}}{E_1'^2} - E_2'^2 - E_3'^2 \;\hat{=}\; \vec{L}^2 \text{ in QM}$$

(QM angular momentum
$\ell(\ell+1)\, |\ell m\rangle = \vec{L}^2 |\ell\, m\rangle$
$m\, |\ell m\rangle = L_z |\ell\, m\rangle$

1. find a basis in which C is diagonal
2. transform E_i' into this basis

$$C' = S^{-1} C S = \mathrm{diag}(\underbrace{6, \dots, 6}_{5\times}, \underbrace{2, \dots, 2}_{3\times}, 0)$$

$$E_i'' = S^{-1} E_i' S = \begin{pmatrix} 5\times 5 & 0 & 0 \\ 0 & 3\times 3 & 0 \\ 0 & 0 & 0_{1\times 1} \end{pmatrix}$$

irreps

Therefore we say :

$3 \times 3 \longrightarrow 1 + 3 + 5$

product representation