8. The Renormalisation Group

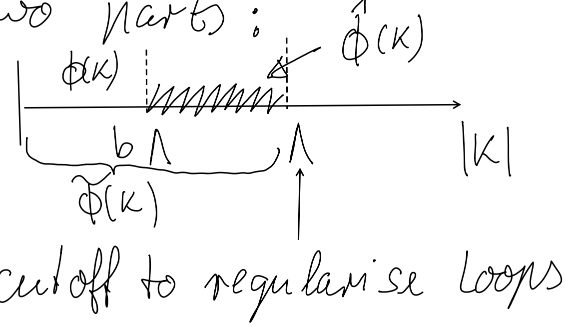
Last lectures: Renormalisation $\hat{=}$ adsorb UV divergences into parameter of bare Lagrangian.

Question: Is there a better physical motivation than just „make observables finite" ?

Idea (by Kenneth Wilson): Split the path integral i.e. for $\tilde{\phi}^4$ theory into two parts:

$$\tilde{\phi}(k) \to \phi(k) + \hat{\phi}(k)$$

$$\hat{\phi}(k) = \begin{cases} \tilde{\phi}(k) & \text{for } b\Lambda < |k| < \Lambda \\ 0 & \text{otherwise} \end{cases}$$



to compute:

$$Z = \int \mathcal{D}\phi \int \mathcal{D}\hat{\phi}\, \exp\left[-\mathcal{L}_E(\phi + \hat{\phi})\right]$$

Wick-rotated Lagrangian, i.e.

$$\mathcal{L}_E(\phi) = \int d^dx \left[\tfrac{1}{2}(\partial_\mu \phi)^2 + \tfrac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!}\phi^4\right]$$

$$Z = \int \mathcal{D}\phi\, e^{-\mathcal{L}_E(\phi)} \int \mathcal{D}\hat{\phi}\, \exp\Big(-\int d^dx \Big[\tfrac{1}{2}(\partial_\mu \hat{\phi})^2 + \tfrac{1}{2} m^2 \hat{\phi}^2 + \lambda\left(\tfrac{1}{6}\phi^3\hat{\phi} + \tfrac{1}{4}\phi^2\hat{\phi}^2 + \tfrac{1}{6}\phi\hat{\phi}^3 + \tfrac{1}{4!}\hat{\phi}^4\right)\Big]\Big)$$

tactic: integrate out momentum $\hat{\phi}$ to obtain

$$= \int [\mathcal{D}\phi]_{b\Lambda} \exp\left(-\int d^dx\, \mathcal{L}_{eff}(\phi)\right)$$

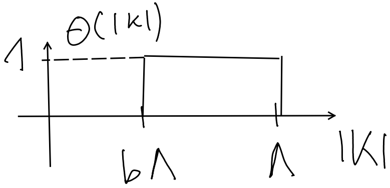
momentum cutoff

$\longrightarrow$ use Feynman diagrams and treat ------ terms as perturbation ($m^2\hat{\phi}^2$ too, because $m^2 \ll \Lambda^2$)

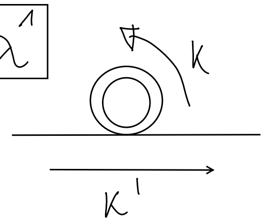
propagator:

$$\underset{k}{\overline{\overline{\quad\quad}}} := \frac{\int \mathcal{D}\hat{\phi}\, e^{-\int \mathcal{L}_0}\, \hat{\phi}(k)\hat{\phi}(p)}{\int \mathcal{D}\hat{\phi}\, e^{-\int \mathcal{L}_0}} = \frac{(2\pi)^d}{k^2}\, \delta^{(d)}(k+p)\, \Theta(k)$$

with $\int \mathcal{L}_0 = \int\limits_{b\Lambda \leq |k| < \Lambda} \frac{d^d k}{(2\pi)^d} \, k^2 |\phi(k)|^2$ and



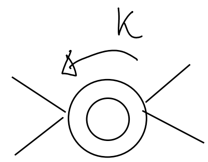
$\boxed{\lambda^1}$



$$= -\tfrac{1}{2} \int \frac{d^d k'}{(2\pi)^d} \mu |\phi(k')|^2 = -\tfrac{1}{2} \int d^d x \, \mu \phi^2$$

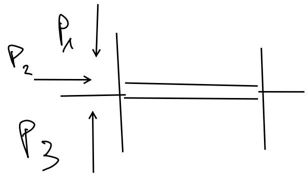
$$\mu = \frac{\lambda}{2} \int\limits_{b\Lambda \leq |k| < \Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} = \frac{\lambda}{(4\pi)^{d/2} \Gamma(d/2)} \frac{1 - b^{d-2}}{d-2} \Lambda^{d-2}$$

$\boxed{\lambda^2}$



$$= -\tfrac{1}{4!} \int d^d x \, \xi \phi^4$$

$$\xi = -\tfrac{3}{2} \lambda^2 \int\limits_{b\Lambda \leq |k| < \Lambda} \frac{d^d k}{(2\pi)^d} \left(\frac{1}{k^2}\right)^2 = \ldots \underset{d \to 4}{=} -\frac{3\lambda^2}{16\pi^2} \log \frac{1}{b}$$



$$\propto \frac{\lambda^2}{(p_1 + p_2 + p_3)^2} \Theta(|p_1 + p_2 + p_3|)$$

and many more ...

$$\longrightarrow \mathcal{L}_{eff} = \frac{1}{2} (\partial'_\mu \phi')^2 + \frac{1}{2} m'^2 \phi'^2 + \frac{1}{4!} \lambda' \phi'^4 + (\text{sum of connected diagrams})$$

Todo: clarify relation between the inital action S_E and the effective action S_{eff}

$$\int d^d x \, \mathcal{L}_{eff} = \int d^d x \left[\frac{1}{2} (1 + \Delta Z)(\partial_\mu \phi)^2 + \frac{1}{2}(m^2 + \Delta m^2) \phi^2 + \frac{1}{4!} (\lambda + \Delta\lambda) \phi^4 + \Delta C (\partial_\mu \phi)^4 + \Delta D \phi^6 + \ldots \right]$$

rescale also $k' = k/b$ and $x' = x \cdot b$

$$= \int d^d x' \, b^{-d} \left[\frac{1}{2} (1 + \Delta Z) b^2 (\partial'_\mu \phi)^2 + \frac{1}{2} (m^2 + \Delta m^2) \phi^2 + \frac{1}{4} (\lambda + \Delta\lambda) \phi^4 + \Delta C \, b^4 (\partial'_\mu \phi)^4 + \Delta D \phi^6 + \ldots \right]$$

to match with $\mathcal{L}_{eff}$ above we have to identify

$$
\left.\begin{aligned}
\phi' &= \left[b^{2-d}(1+\Delta Z)\right]^{1/2}\phi, \\
m'^2 &= (m^2+\Delta m^2)(1+\Delta Z)^{-1} b^{-2}, \\
\lambda' &= (\lambda+\Delta\lambda)(1+\Delta Z)^{-2} b^{d-4}, \\
C' &= (C+\Delta C)(1+\Delta Z)^{-2} b^{d}, \quad \text{and} \\
D' &= (D+\Delta D)(1+\Delta Z)^{-3} b^{2d-6}
\end{aligned}\right\}
\begin{array}{l} \Delta Z,\ \Delta m^2, \dots \\ \text{come from the} \\ \text{diagrams we} \\ \text{just computed} \end{array}
$$

We are back at the point where we started and can now integrate out another momentum shell. For $b \to 1$ the transformation becomes continuous and is called <u>renormalisation group flow</u>.

8.1. Renormalisation Group Flow

Example: $\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)^2$ is left unchanged by the flow because $\Delta m^2, \Delta\lambda, \dots$ all vanish.

$\longrightarrow$ <u>fixed point</u>. Close to it, we find

$$m'^2 = m^2 b^{-2}, \quad \lambda' = \lambda b^{d-4}, \quad C' = C b^{d}, \quad D' = D b^{2d-6}, \quad \dots$$

$$-N\left(\tfrac{d}{2}-1\right) - M + d \qquad \text{for} \quad \begin{array}{l} N: \text{powers of } \phi \\ M: \#\,\text{derivatives} \end{array}$$

$$= d - \underbrace{d_{N,M}}_{\text{mass dimension}}$$

$$\boxed{\text{UV}: \mathcal{L}_0} \xrightarrow{\text{RG flow towards}} \boxed{\text{IR}: \mathcal{L}_{eff}}$$

because $b<1$ operators are

$$
\left.\begin{aligned}
d_{N,M} &< d: && \text{relevant} \\
d_{N,M} &> d: && \text{irrelevant} \\
d_{N,M} &= 0: && \underline{\underline{\text{marginal}}}
\end{aligned}\right\}
\begin{array}{l} \text{Same as the analysis} \\ \text{of superficial degree of} \\ \text{divergence in section 7.1} \end{array}
$$

$\Delta m^2, \Delta\lambda, \dots$ can change the flow

i.e. in $d=4$ $\quad \lambda' = \lambda - \frac{3\lambda^2}{16\pi^2}\log(1/b)$

→ marginally irrelevant