An Introduction to String Theory, Winter 2022/23

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7. Open string in light-cone gauge (22 points)

To be discussed on Thursday, 24th November, 2022 in the tutorial.
Please indicate your preferences until Saturday, 19/11/2022, 21:00:00 on the website.

We reached the halftime of the course and in this exercise, you can revisit a lot of concepts we learned until now. Thus, even if you are assigned just to certain tasks, it might be good to try to solve both problems completely and check how much from the material discussed in the lecture you understand. If you struggle at any point, keep in mind: Light-cone quantisation is one of the most basic approaches to string theory. There it is contained in nearly any textbook on the subject.

Exercise 7.1: The classical story

In this problem, we want to study the classical open string in light-cone gauge which is given by

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^1) \quad \text{and} \quad X^I = X^I \quad I = 2, \dots, d\,.$$

a) (1 point) Verify the results

$$-(\mathrm{d}X^0)^2 + \sum_{i=1}^{d} (\mathrm{d}X^i)^2 = -2\mathrm{d}X^+\mathrm{d}X^- + \sum_{I=2}^{d} (\mathrm{d}X^I)^2$$

from the lecture and read-off the form of the Minkowski metric $\eta_{\mu\nu}$ in light-cone coordinates.

b) (2 points) Form exercise 5.1) we remember that the most general solutions for the two-dimensional wave equation that governs the dynamics of the string is

$$X^\mu(\sigma, \tau) = \frac{1}{2}\left(X_R(\tau - \sigma) + X_L(\tau + \sigma)\right)$$

First, show that the Dirichlet boundary conditions

$$X'(\tau, 0) = X'(\tau, \pi) = 0$$

implies

$$X'_L(u) = X'_R(u) \qquad \text{and}$$
$$X'_L(u) = X'_L(u + 2\pi)\,.$$

Therefore any solution which is compatible with these boundary conditions is completely captured by the periodic function $X'_L(\sigma)$ and its integration constants. Integrate

$$X'_L(\sigma) = \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} e^{-in\sigma} \alpha_n$$

to compute the mode expansion of $X^\mu(\tau, \sigma)$.

Now we impose light-cone gauge

$$X^+(\sigma, \tau) = 2p^+\alpha'\tau.$$

c) (2 points) Check that the constraint equations $(\dot{X} \pm X')^2 = 0$ reduce in this gauge to

$$X'_L X'_L = -2\alpha' p^+ X'^-_L + X'^I_L X'_{LI} = 0\,.$$

The remarkable thing about this results (and light-cone gauge) is that X'^-_L appears linearly and we can easily solve for it. Compute $X^-(\tau, \sigma)$ by integrating the result.

d) (2 points) We now want to learn more about the Fourier modes that describe $X^-(\sigma)$, which we obtained in the previous task. Introduce the transverse Virasoro modes

$$L_n^\perp = \frac{1}{2}\sum_p \alpha_p^I \alpha_{(n-p)I}$$

to show

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha' p^+}} L_n^\perp\,.$$

In particular interesting is $L_0^\perp$, because we need it to compute the Hamiltonian and from it the mass of string excitations. Assume

$$\alpha_0^- = \sqrt{2\alpha'} p^-$$

and verify

$$L_0^\perp = \alpha' p^I p_I + \underbrace{\sum_{p=1}^{\infty} |\alpha_p^\perp|^2}_{N^\perp}\,.$$

e) (2 points) Start from the Lagrangian

$$L = \frac{1}{4\pi\alpha'}\left(\dot{X}^2 - X'^2\right)$$

and compute its canonical momentum $\Pi_\mu(\tau, \sigma)$ and Hamiltonian H.

Hint: We already did this in the lecture. It is a quick calculation. Just review it here and be careful with all the signs and prefactors.

Next, derive the Poisson brackets for the coefficients α_n^μ and x^μ you found in the mode expansion for $X^\mu(\tau, \sigma)$ in b).

Hint: Find out first the Fourier expansion of the δ-function in the canonical Poisson brackets

$$\{\Pi_\mu(\tau, \sigma), X^\nu(\tau, \sigma)\} = \delta_\mu^\nu \delta(\sigma - \sigma')\,.$$

f) (2 points) Use the result from the previous tasks above to verify the mass formula

$$M^2 = 2p^+ p^- - p^{\perp 2} = \frac{1}{\alpha'} N^\perp\,. \tag{1}$$

Exercise 7.2: Light-cone quantisation

After understanding the classical regime (see problem 1), we can proceed with quantising the open string in light-cone gauge.

a) (2 points) Collect all the fundamental degrees of freedom we identified in problem 1, and their Poisson brackets to obtain all relevant commutation relation by canonical quantisation

$$\{\cdot,\, \cdot\} \to -i[\cdot,\, \cdot]\,.$$

b) (1 point) Describe the resulting Hilbert space $\mathcal{H}_{LCG}$. Write down the level one state which corresponds to a spacetime vector.

Hint: The only unconstrained raising/lowering operators are $\alpha^I_{-n}/\alpha^I_n = (\alpha^I_{-n})^\dagger$ *for* $n > 0$.

To calculate the mass of this state, we have to add a normal ordering constant a to the mass formula (1), resulting in

$$M^2 = \frac{1}{\alpha'}\left(N^\perp - a\right)\,.$$

We now try to compute this constant, by what is called ζ-function regularisation. The idea here is to regularise the sum

$$-\frac{D-2}{2}\sum_{p=1}^{\infty} p = a\,,$$

which arises from the normal ordering. To this end, we note that the Riemann ζ-function is given by

$$\zeta(s) = \sum_{p=1}^{\infty} p^{-s}\,.$$

c) (2 points) Prove that we can write the product of the ζ- and Γ-function,

$$\Gamma(s) = \int_0^\infty \mathrm{d}t\, e^{-t} t^{s-1}\,,$$

as

$$\Gamma(s)\zeta(s) = \int_0^\infty \mathrm{d}t \frac{t^{s-1}}{e^t - 1}$$

assuming that Re(s)>1.

d) (1 point) Verify the small t expansion

$$\frac{1}{e^t - 1} = \frac{1}{t} - \frac{1}{2} + \frac{t}{12} + \mathcal{O}(t^2)\,,$$

e) (1 point) and use it to show that for Re(s)>1

$$\Gamma(s)\zeta(s) = \int_0^1 \mathrm{d}t\, t^{s-1}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12}\right) + \frac{1}{s-1} + \frac{1}{2s} + \frac{1}{12(s+1)} + \int_1^\infty \mathrm{d}t \frac{t^{s-1}}{e^t - 1}$$

holds.

f) (1 point) Explain why the right-hand side above is well defined also for Re(s)>-2. It follows that this right-hand side defines an analytic continuation of the left-hand side to Re(s)>-2.

g) (2 points) Recall the pole structure of the Γ-function and use it to show that

$$\zeta(0) = -\frac{1}{2} \qquad \text{and} \qquad \zeta(-1) = -\frac{1}{12}\,.$$

Argue that the ζ-function regularisation implies that

$$\sum_{p=1}^{\infty} p = -\frac{1}{12}. \tag{2}$$

h) (1 point) Explain, why an open string with a massless spacetime vector implies $D = 26$.